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OLAVI VUORELAINEN

The temperature field produced in the ground by a heated slab laid direct on ground, and the heat flow from slab to ground

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THE TEMPERATURE FIELD PRODUCED IN THE GROUND BY A HEATED SLAB LAID DIRECT ON GROUND, AND THE HEAT FLOW FROM SLAB TO GROUND

OLAVI VUORELAINEN

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INTRODUCTION

Widely different conditions may exist in the ground under buildings erected on a concrete slab, depending on the characteristics of the building site. It is therefore difficult to derive universally valid rules relating to the temperatures in the ground and to the heat quantities lost through flow into the ground merely on the basis of measurements. Mathematical equations have to be set up, which are generally valid under stipulated boundary conditions. Comparison between the calculated values and the results of measurements carried out in experimental houses enables rules applicable on a wider basis to be inferred from experimental studies made with a few houses only.

Various theoretical formulae and formulae based on measurements have been developed for the purpose of calculating the heat losses through a floor slab. Among them, the method rather widely employed in America may be mentioned, in which only the losses at the edges are taken into account. The heat quantity given off by the slab to the ground is calculated by the equation

$$Q = C(T_i - T_o) \cdot (2a + 2b),$$

where C is a value determined experimentally for various types of structures, $T_i - T_o$ is the temperature difference between outdoor and indoor air, and $2(a + b)$ is the circumference of the building [1].

Owing to the entirely different kind of climate and soil types, these experimental formulae cannot be used as such in Finnish conditions.

Billington divides the losses into two components by means of an electric analogy method: (1) the edge losses, which consist of the heat passing through the upper soil layers into ambient air, and (2) the surface losses, i.e., the heat quantity flowing from the central parts of the slab into soil layers at greater depth. Both

components are calculated separately in Billington's method by a simple computing procedure, assuming that the isothermal lines are ellipses [2].

Ruckli has devised a calculating method for round slabs, from which the results can be transferred to rectangular slabs with the aid of a correction coefficient. This method is primarily intended for use in the designing of cold-storage rooms [3].

The methods just mentioned yield mutually different results because they employ approximative formulae.

O. Krischer has presented an approximative method for calculating the heat losses going into the ground from buildings erected direct on ground [4]; it has been modified with a view to more convenient use by W. Weyh [5]. Krischer calculated the stationary thermal flow from a rectangular building into the ground when the constant temperatures T_1 and T_2 prevail inside and outside and the ground temperature is constant at the depth $Z = Z_0$. His solution, derived with the aid of series expansions, requires that the heat transfer coefficient (α) be known. However, the heat transfer coefficients are dependent on air flow velocity (i.e, on the wind) and therefore cannot be considered constant.

In the present work, the equations of the temperature distribution field and the thermal flow conveyed into the ground under stationary conditions are derived for a rectangular slab, for a slab shaped as a narrow strip (terrace houses), and for a circular foundation slab. These equations are set up for various boundary conditions. In order to avoid excessively complicated equations, the said equations in the case of non-stationary thermal flow are given only for a narrow strip slab. They are given for the following alternatives: (a) The temperature of the slab is a given function of time, T(t), (b) the thermal flow from the slab into the ground is a given function of time, Q(t).

The temperature of the slab and its variations in terms of time, as well as the thermal flow conveyed from the slab to the ground, are obtained from measurements. Selecting the temperature function on the lower surface of the slab and on its edges in close correspondence with actual conditions, one obtains results conforming with measurements, both with regard to the temperature field and the thermal flow.

THE TEMPERATURE FIELD PRODUCED IN THE GROUND BY A HEATED SLAB

The temperature in the ground at different depths, $T_1(z,t)$ goes through seasonal variations in accordance with the solution of a one-dimensional equation [6]. This temperature distribution field accountable to the changes of outdoor temperature is superimposed upon the temperature field $T_2(x,y,z,t)$ produced by the slab. When the thermal power $Q(x,y,t)$ flowing from the slab into the ground is known, the temperature field derived from this information can be immediately added to the inherent temperature field of the ground, point by point, the resulting, actual temperature field being

$$T(x,y,z,t) = T_1\ (z,t) + T_2\ (x,y,z,t)$$

In the case that the temperature of the slab, $T(t)$, which is assumed to be constant with respect to the position coordinates, is known as a result of measurements, this temperature $T(t)$ already includes the influence of the ground temperature field. In calculating the field $T_2(x,y,z,t)$ produced by the slab, which is superimposed by addition on the ground field, the difference between the actual temperature of the slab and the ground surface temperature, $T(t) - T_1(0,t)$, has to be used for the surface temperature of the slab in the boundary conditions.

STATIONARY FIELD

When the ground surface is covered by a snow layer for a length of time and the variations of the outdoor temperature are very small, the temperature field in the ground can be considered approximately stationary, $T_1 = T_1(z)$, also in the upper layers. A slab which does not act as a heat source but obtains its heat supply from the room air remains at approximately constant temperature throughout the year, and the temperature field under the slab displays rather small variations only. Also the field produced by a slab acting as a heat source attains a stationary state $T_2 = T_2(x, y, z)$ when the outdoor air temperature is constant during the time of about one month.

The ultimate field in the stationary state can be described by means of the equation

$$T = T_1(z) + T_2(x, y, z),$$

In the stationary state, both functions appearing in this equation satisfy Laplace's equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{1}$$

RECTANGULAR FOUNDATION SLAB

The temperature outside the slab decreases in accordance with a step function, from T_0 to 0.

The function $T_2(x, y, z)$ representing the temperature in the ground, which will be written without its sub-index in the following, shall satisfy the differential equation (1) and the following boundary conditions:

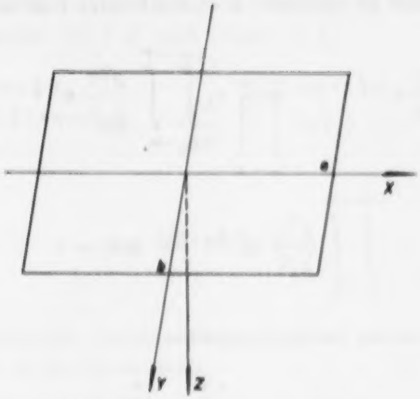

Fig. 1. The choice of coordinates for a rectangular foundation slab.

$$
\begin{aligned}
&\text{a)}\quad T(x,y,0) = T_0, && \text{when } -a < x < a \text{ and } -b < y < b \\
&\text{b)}\quad T(x,y,0) = 0, && \text{when } -a > x > a \text{ and } -b > y > b \\
&\text{c)}\quad \frac{\partial T(x,y,0)}{\partial z} = 0, && \text{when } -a > x > a \text{ and } -b > y > b \\
&\text{d)}\quad \text{For } Z \to \infty, \quad T \to 0
\end{aligned}
\tag{2}
$$

The Fourier double transform of the temperature function

$$T^{*}(\xi,\eta,z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} T(x,y,z)\, e^{i(\xi x+\eta y)}\, dxdy \tag{3}$$

satisfies the differential equation (1) of the temperature distribution. The object function $T(x,y,z)$ is obtained in accordance with the inversion theorem of the Fourier transformation [7],

$$T(x,y,z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} T^{*}(\xi,\eta,z)\, e^{-i(\xi x+\eta y)}\, d\xi d\eta \tag{4}$$

Subjecting each term in equation (1) to the Fourier transformation

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\partial^2 T}{\partial x^2} e^{i(\xi x+\eta y)}\, dxdy + \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\partial^2 T}{\partial y^2} e^{i(\xi x+\eta y)}\, dxdy +$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\partial^2 T}{\partial z^2} e^{i(\xi x+\eta y)}\, dxdy = 0$$

and expanding the terms by partial integration

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\partial^2 T}{\partial x^2} e^{i(\xi x+\eta y)}\, dxdy = \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) e^{i(\xi x+\eta y)}\, dx;$$

$$\int_{-\infty}^{+\infty} \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) e^{i(\xi x+\eta y)}\, dx = \Big/_{-\infty}^{+\infty} \frac{\partial T}{\partial x} e^{i(\xi x+\eta y)} - i\xi \int_{-\infty}^{+\infty} \frac{\partial T}{\partial x} e^{i(\xi x+\eta y)}\, dx$$

we find upon application of the boundary conditions (2a) and (2b)

$$0 - i\xi \int_{-\infty}^{+\infty} \frac{\partial T}{\partial x} e^{i(\xi x+\eta y)}\, dx = 0 - i\xi \Big/_{-\infty}^{+\infty} T\, e^{i(\xi x+\eta y)} - \xi^2 \int_{-\infty}^{+\infty} T\, e^{i(\xi x+\eta y)}\, dx =$$

$$0 + 0 - \xi^2 \int_{-\infty}^{+\infty} T(x, y, z)\, e^{i(\xi x+\eta y)}\, dx$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\partial^2 T}{\partial x^2} e^{i(\xi x+\eta y)}\, dxdy = -2\pi\ \xi^2\ T^*(z, \xi, \eta)$$

The second integral of the expression (1), treated in the same manner, gives

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\partial^2 T}{\partial y^2} e^{i(\xi x+\eta y)}\, dxdy = -2\pi\ \eta^2\ T^*(\xi, \eta, z)$$

The third integral in (1) is the second derivative of the Fourier transform of the temperature function T(x, y, z) with respect to z,

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\partial^2 T}{\partial z^2} e^{i(\xi x+\eta y)} \, dxdy = \frac{\partial^2}{\partial z^2} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} T(x,y,z)\, e^{i(\xi x+\eta y)} \, dxdy =$$

$$2\pi \frac{\partial^2}{\partial z^2} T^*(\xi, \eta, z)$$

By these transformations, equation (1) is converted into a linear, homogeneous differential equation of the second order

$$\frac{\partial^2 T^*}{\partial z^2} - (\xi^2 + \eta^2)\, T^* = 0 \tag{5}$$

The general solution of this equation is [9]

$$T^*(\xi, \eta, z) = C_1(\xi, \eta)\, e^{-\sqrt{\xi^2 + \eta^2}\, Z} + C_2(\xi, \eta)\, e^{+\sqrt{\xi^2 + \eta^2}\, Z} \tag{6}$$

Only the first term comes into question because the second term increases infinitely with increasing Z. When $C_2(\xi, \eta) \equiv 0$,

$$T^*(\xi, \eta, z) = C_1(\xi, \eta)\, e^{-\sqrt{\xi^2 + \eta^2}\, Z} \tag{7}$$

We substitute the expression (7) of $T^*(\xi, \eta, z)$ in the equation (4), representing the temperature distribution, taking into consideration that, according to Euler's formula [8],

$$e^{i(\xi x+\eta y)} = \cos(\xi x+\eta y) + i \sin(\xi x+\eta y)$$

In this manner we obtain the expression for the temperature field in the form of Fourier's cosine transform

$$T(x,y,z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} C_1(\xi, \eta)\, e^{-\sqrt{\xi^2 + \eta^2}\, Z} \cos(\xi x+\eta y)\, d\xi\, d\eta \tag{8}$$

when only the real part is considered, for obvious reasons of symmetry.

The unknown function $C_1(\xi,\eta)$ in the integrand is determined so that the expression of $T(x,y,z)$ satisfies the boundary conditions (2a) and (2b). According to equation (6) and (3) and to the boundary condition (2b),

$$T^*(\xi,\eta,0) = C_1(\xi,\eta) = \frac{1}{2\pi}\int_{-a}^{+a}\int_{-b}^{+b} T(x,y,0)\cos(\xi x+\eta y)\,dxdy =$$

$$\frac{T_o}{2\pi}\int_{-a}^{+a}\int_{-b}^{+b} \cos(\xi x+\eta y)\,dxdy \tag{9}$$

Integration yields further

$$T^*(\xi,\eta,0) = \frac{2T_o}{\pi}\,\frac{\sin\xi a\,\sin\eta b}{\xi\eta} \tag{10}$$

On the strength of equations (9) and (10), the function $T(x,y,z)$, equation (8), describing the temperature distribution acquires the form

$$T(x,y,z) = \frac{T_o}{\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\sin\xi a\,\sin\eta b}{\xi\eta}\,e^{-\sqrt{\xi^2+\eta^2}\,z}\cos(\xi x+\eta y)\,d\xi d\eta \tag{11}$$

In order that dimensionless quantities might be used, a and b are chosen as units of measurement, that is, we write $\xi a = \alpha$ and $\eta b = \beta$. Bearing symmetry in mind at the same time, we now obtain equation (11) in the form

$$T(x,y,z) = \frac{4T_o}{\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\sin\alpha}{\alpha}\,\frac{\sin\beta}{\beta}\,e^{-z\sqrt{\left(\frac{\alpha}{a}\right)^2+\left(\frac{\beta}{b}\right)^2}}\cos\left(\frac{x\alpha}{a}+\frac{y\beta}{b}\right)d\alpha d\beta \tag{12}$$

Calculating by equation (11), the thermal flow from the slab to the ground, per element of surface area, is

$$dQ = -\lambda\,\frac{\partial T(x,y,0)}{\partial z}\,dxdy \tag{13}$$

and the total thermal flow in the region affected by the slab is

$$Q = \frac{16T_o\lambda}{\pi^2}\int_0^{\infty}\int_0^{\infty}\left(\frac{\sin\xi a}{\xi}\right)^2\left(\frac{\sin\xi b}{\eta}\right)^2\sqrt{\xi^2+\eta^2}\,d\xi d\eta \tag{14}$$

When calculated by equation (11), the temperature gradient obtains the value ∞ at the edge of the slab, and consequently the thermal flow will also be infinite on the edge. In this case also the total thermal flow from slab to ground becomes infinite, as can be seen from its expression (14).

On physical interpretation, this is found to state merely a self-evident fact: heat would have to be supplied to the slab at an infinitely high rate in order to keep its temperature constant up to its very edge when the surroundings of the slab are at 0^oC.

This shortcoming of the theory can be corrected by choosing new boundary conditions, assuming that the temperature of the slab in its marginal parts decreases linearly from T_o to zero over a given distance, which depends on the design of the slab. This approximation is well consistent with actual conditions encountered in practice: owing to the heat release of melting snow and to good insulating properties of snow, the ground temperature under the snow cover adjacent to the edge of the slab, and also at greater distance, maintains a height close to 0^oC.

The temperature in the marginal part of the slab decreases linearly from T_o to 0.

The new boundary conditions (15) have been chosen on the basis of experience gained in connection with actual measurements.

$$
\begin{array}{lll}
\text{a)}\ T(x,y,0) = T_o & \text{when} & -a < x < a \quad \text{and} \quad -b < y < b \\
\text{b)}\ T(x,y,0) = T_o - \dfrac{(x-a)\,T_o}{a'-a} & \text{"} & a < x < a' \\
\text{c)}\ T(x,y,0) = T_o - \dfrac{(y-b)\,T_o}{b'-b} & \text{"} & b < y < b' \\
\text{d)}\ T(x,y,0) = 0 & \text{"} & x > a' \\
\text{e)}\ T(x,y,0) = 0 & \text{"} & y > b'
\end{array}
\qquad (15)
$$

In these formulae, $-\frac{T_o}{a'-a}$ and $-\frac{T_o}{b'-b}$ represent the rate of temperature drop in the x and y directions, respectively, at the edges of the slab.

We shall now determine the unknown function $C_1(\xi,\eta)$ in equation (8) in compliance with the new boundary conditions (15)

$$C_1(\xi,\eta) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} T(x,y,0) \cos(\xi x + \eta y)\,dx\,dy \qquad (16)$$

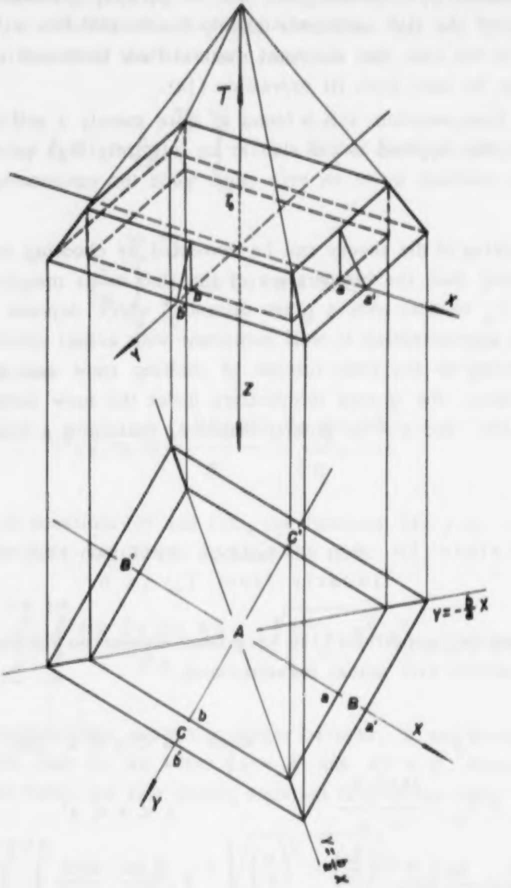

Fig. 2. Rectangular foundation slab; its temperature at the edge decreasing linearly from T_o to zero.

The Fourier transform of the temperature function on the surface $z = 0$ is calculated as a surface integral, separately for the regions A, B, B', C and C'.

$$I_A = T_o \int_{-a}^{a} \int_{-b}^{b} \cos(\xi x + \eta y)\,dx\,dy = 4T_o \frac{\sin \xi a \, \sin \eta b}{\xi \eta}$$

$$I_B = \int_a^{a'} \int_{-\frac{b}{a}x}^{\frac{b}{a}x} \left[T_o - \frac{(x-a)\ T_o}{a'-a} \right] \cos(\xi x + \eta y) dxdy \tag{17}$$

$$I_C = \int_b^{b'} \int_{-\frac{a}{b}y}^{\frac{a}{b}y} \left[T_o - \frac{(y-b)\ T_o}{b'-b} \right] \cos(\xi x + \eta y) dydx$$

$$I_D = 0$$

$$I_E = 0$$

$$I_B = I_{B_1} + I_{B_2} = \left[T_o + \frac{aT_o}{a'-a} \right] \int_a^{a'} \int_{-\frac{b}{a}x}^{\frac{b}{a}x} \cos(\xi x + \eta y) dxdy - \frac{T_o}{a'-a} \int_a^{a'} \int_{-\frac{b}{a}x}^{\frac{b}{a}x} x \cos(\xi x + \eta y) dxdy \tag{18}$$

$$I_C = I_{C_1} + I_{C_2} = \left[T_o + \frac{bT_o}{b'-b} \right] \int_b^{b'} \int_{-\frac{a}{b}y}^{\frac{a}{b}y} \cos(\xi x + \eta y) dxdy - \frac{T_o}{b'-b} \int_b^{b'} \int_{-\frac{a}{b}y}^{\frac{a}{b}y} y \cos(\xi x + \eta y)\ dxdy \tag{19}$$

$$I_{B_1} = \frac{a' T_o}{a' - a} \int_a^{a'} \int_{-\frac{b}{a}x}^{\frac{b}{a}x} \cos(\xi x + \eta y) dxdy = \frac{a' T_o}{a' - a} \left[\frac{-\cos(a'\xi + b'\eta) + \cos(a\xi + b\eta)}{(\xi + \frac{b}{a}\eta)\eta} \right.$$

$$\left. \frac{\cos(a'\xi - b'\eta) - \cos(a\xi - b\eta)}{(\xi - \frac{b}{a}\eta)\eta} \right] \tag{20}$$

$$I_{B_2} = -\frac{T_o}{a'-a}\int_a^{a'}\int_{-\frac{b}{a}x}^{\frac{b}{a}x} x\cos(\xi x+\eta y)\,dx\,dy = \frac{T_o}{a'-a}\left[\frac{-a'\cos(a'\xi+b'\eta)+a\cos(a\xi+b\eta)}{(\xi+\frac{b}{a}\eta)\eta} + \right.$$

$$\frac{a'\cos(a'\xi+b'\eta)-a\cos(a\xi-b\eta)}{(\xi-\frac{b}{a}\eta)\eta} + \frac{\sin(a'\xi+b'\eta)-\sin(a\xi+b\eta)}{(\xi+\frac{b}{a}\eta)^2\eta} -$$

$$\left.\frac{\sin(a'\xi-b'\eta)-\sin(a\xi-b\eta)}{(\xi-\frac{b}{a}\eta)^2\eta}\right] \tag{21}$$

From equations (20) and (21), we find

$$I_B = I_{B_1} + I_{B_2} = T_o\left[\frac{a\cos(a\xi+b\eta)}{(a\xi+b\eta)\eta} - \frac{a\cos(a\xi-b\eta)}{(a\xi-b\eta)\eta} - \right.$$

$$\left.\frac{a^2}{a'-a}\,\frac{\sin(a'\xi+b'\eta)-\sin(a\xi+b\eta)}{(a\xi+b\eta)^2\eta} + \frac{a^2}{a'-a}\,\frac{\sin(a'\xi-b'\eta)-\sin(a\xi-b\eta)}{(a\xi-b\eta)^2\eta}\right] \tag{22}$$

The integral $I_C = I_{C_1} + I_{C_2}$ (equation 15c) is obtained in a similar manner

$$I_C = T_o\left[\frac{b\cos(a\xi+b\eta)}{(a\xi+b\eta)\xi} + \frac{b\cos(a\xi-b\eta)}{(a\xi-b\eta)\xi} - \frac{b^2}{b'-b}\,\frac{\sin(a'\xi+b'\eta)-\sin(a\xi+b\eta)}{(a\xi+b\eta)^2\xi} - \right.$$

$$\left.\frac{b^2}{b'-b}\,\frac{\sin(a'\xi-b'\eta)-\sin(a\xi-b\eta)}{(a\xi-b\eta)^2\xi}\right] \tag{23}$$

The surface integral for the combination of regions B and C is derived from equations (22) and (23)

$$I_B + I_C = T_o\left[\frac{\cos(a\xi+b\eta)-\cos(a\xi-b\eta)}{\xi\eta} - \right.$$

$$\frac{\left(a^2\xi(b'-b)+b^2\eta(a'-a)\right)\left(\sin(a'\xi+b'\eta)-\sin(a\xi+b\eta)\right)}{(a'-a)(b'-b)(a\xi+b\eta)^2\xi\eta} +$$

$$\left.\frac{\left(a^2\xi(b'-b)-b^2\eta(a'-a)\right)\left(\sin(a'\xi-b'\eta)-\sin(a\xi-b\eta)\right)}{(a'-a)(b'-b)(a\xi-b\eta)^2\xi\eta}\right] \tag{24}$$

For the entire region $-a' < x < a'$; $-b' < y < b'$, over which the heating influence of the slab at ground surface extends, there is according to equations (17)

$$I = I_A + 2I_B + I_C + I_D + I_E = 2T_o\left[\left(\frac{a^2\xi}{a'-a} - \frac{b^2\eta}{b'-b}\right)\frac{\sin(a'\xi - b'\eta) - \sin(a\xi - b\eta)}{(a\xi - b\eta)^2\xi\eta} - \right.$$

$$\left.\left(\frac{a^2\xi}{a'-a} + \frac{b^2\eta}{b'-b}\right)\frac{\sin(a'\xi + b'\eta) - \sin(a\xi + b\eta)}{(a\xi + b\eta)^2\xi\eta}\right] \qquad (25)$$

As can be seen from Fig. 2, $\frac{a}{a'-a} = \frac{b}{b'-b}$. Consequently, the expression in angular brackets can be simplified and equation (16) becomes

$$C_1(\xi,\eta) = \frac{T_o}{\pi}\frac{a}{a'-a}\left[\frac{\sin(a'\xi - b'\eta) - \sin(a\xi - b\eta)}{(a\xi - b\eta)\xi\eta} - \frac{\sin(a'\xi + b'\eta) - \sin(a\xi + b\eta)}{(a\xi + b\eta)\xi\eta}\right]$$

The equation of the temperature distribution field is obtained from equations (8) and (25)

$$T(x,y,z) = \frac{T_o}{2\pi^2}\frac{a}{a'-a}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[\frac{\sin(a'\xi - b'\eta) - \sin(a\xi - b\eta)}{(a\xi - b\eta)\xi\eta} - \right. \qquad (26)$$

$$\left.\frac{\sin(a'\xi + b'\eta) - \sin(a\xi + b\eta)}{(a\xi + b\eta)\xi\eta}\right]e^{-z\sqrt{\xi^2 + \eta^2}}\cos(\xi x + \eta y)\,d\xi d\eta$$

Let us write

$$a\xi = \alpha$$
$$b\eta = \beta$$
$$\frac{b}{a} = \frac{b'}{a'} \quad \text{and hence} \quad a'\xi = \frac{b'}{b}\alpha$$
$$b'\eta = \frac{a'}{a}\beta$$

by which the solution can be presented in the following form, which is more appropriate for numerical calculations

$$T(x,y,z) = \frac{2T_o a}{\pi^2(a'-a)}\int_0^{+\infty}\int_0^{+\infty}\left[\frac{\sin\frac{a'}{a}(\alpha - \beta) - \sin(\alpha - \beta)}{(\alpha - \beta)\alpha\beta} - \right.$$

$$\left.\frac{\sin\frac{a'}{a}(\alpha + \beta) - \sin(\alpha + \beta)}{(\alpha + \beta)\alpha\beta}\right]e^{-\frac{z}{a}\alpha\sqrt{1 + \left(\frac{a\beta}{b\alpha}\right)^2}}\cos\frac{x}{a}\alpha \cdot \cos\frac{y}{b}\beta\,d\alpha d\beta \qquad (27)$$

This equation can be employed to calculate the temperature field in any point (x, y, z) under a rectangular floor slab with the sides a and b, produced by this slab in the stationary state.

In principle, the calculation can be carried out by giving β, at first, a fixed value β_ν and performing numerical integration with respect to α, using tables [10]. All values of β are gone through in this manner, the result being that $\int_0^\infty F(\alpha\beta_\nu)\,d\alpha$ is obtained as a function of β. Subsequently, graphical integration with respect to β gives the temperature in the desired point (x, y, z).

Even with the aid of tables, the calculation involves a very great amount of work and it is virtually necessary to resort to a computer. (The IBM 710 computer was used in the present instance.)

We are now going to show that, for $b \to \infty$, the temperature function (27) approaches as its limit the solution (45), p. 25, obtained for a slab having the shape of a narrow strip.

Let I_1 denote the inner integral with β as its integration variable

$$I_1 = \int_{-\infty}^{+\infty} \left[\frac{\sin\frac{a'}{a}(\alpha-\beta) - \sin(\alpha-\beta)}{(\alpha-\beta)\alpha\beta} - \frac{\sin\frac{a'}{a}(\alpha+\beta) - \sin(\alpha+\beta)}{(\alpha+\beta)\alpha\beta} \right] e^{-\frac{z}{a}\alpha\sqrt{1+\left(\frac{a\beta}{b\alpha}\right)^2}} \cos\left(\frac{x}{a}\alpha + \frac{y}{b}\beta\right) d\beta$$

$$\lim \cos\left(\frac{x}{a}\alpha + \frac{y}{b}\beta\right) = \cos\frac{x}{a}\alpha$$

when $b \to \infty$

$$\lim e^{-\frac{z}{a}\alpha\sqrt{1+\left(\frac{a\beta}{b\alpha}\right)^2}} = e^{-\frac{z}{a}\alpha}$$

when $b \to \infty$

As its limiting value, the integral I_1 obtains the form $I_2 = I_1' + I_2' + I_3' + I_4'$

$$I_2 = \int_{-\infty}^{+\infty} \left[\frac{\sin\frac{a'}{a}(\alpha-\beta)}{(\alpha-\beta)\alpha\beta} - \frac{\sin(\alpha-\beta)}{(\alpha-\beta)\alpha\beta} - \frac{\sin\frac{a'}{a}(\alpha+\beta)}{(\alpha+\beta)\alpha\beta} + \frac{\sin(\alpha+\beta)}{(\alpha+\beta)\alpha\beta} \right] d\beta$$

Each integral in the expression of I_2 is calculated separately, the results being

$$I_1' = \int_{-\infty}^{+\infty} \frac{\sin\frac{a'}{a}(\alpha-\beta)}{(\alpha-\beta)\alpha\beta}\, d\beta = \frac{\pi}{\alpha^2} - \frac{\pi\cos\frac{a'}{a}\alpha}{\alpha^2}$$

$$I_2' = \int_{-\infty}^{+\infty} \frac{\sin(\alpha-\beta)}{(\alpha-\beta)\alpha\beta}\, d\beta = \frac{\pi}{\alpha^2} - \frac{\pi\cos\alpha}{\alpha^2}$$

$$I_3' = \int_{-\infty}^{+\infty} \frac{\sin \frac{a'}{a} (\alpha+\beta)}{(\alpha+\beta)\alpha\beta} d\beta = -\frac{\pi}{\alpha^2} + \frac{\pi \cos \frac{a'}{a}\alpha}{\alpha^2}$$

$$I_4' = \int_{-\infty}^{+\infty} \frac{\sin (\alpha+\beta)}{(\alpha+\beta)\alpha\beta} d\beta = -\frac{\pi}{\alpha^2} + \frac{\pi \cos \alpha}{\alpha^2}$$

Summation of the four integral renders

$$I_2 = I_1' + I_2' + I_3' + I_4' = \frac{4\pi}{\alpha^2}\left[\cos \alpha - \cos \frac{a'}{a}\alpha\right]$$

and hence further

$$I_1 = \frac{4\pi}{\alpha^2}\left[\cos \alpha - \cos \frac{a'}{a}\alpha\right] e^{-\frac{z}{a}\alpha} \cos \frac{x}{a}\alpha$$

The limit of the temperature function (27) for $b \to \infty$ will thus be

$$T(x,y) = -\frac{2a\ T_0}{\pi(a'-a)} \int_{-\infty}^{+\infty} \frac{\cos \frac{a'}{a} - \cos \alpha}{\alpha^2}\, e^{-\frac{z}{a}\alpha} \cos \frac{x}{a}\alpha\ d\alpha, \qquad (28)$$

which is the same as equation (45), p. 25 ($a' = b$).

The thermal flow from a rectangular slab to the ground. In order to maintain in the ground under the slab a stationary temperature field in accordance with equation (26), the heat quantity

$$dQ = -\lambda \frac{\partial T(x,y,0)}{\partial z} dxdy \qquad (29)$$

per surface area element has to be conveyed from the slab to the ground. The temperature gradient on the lower surface of the slab is obtained from equation (26) by derivation with respect to z

$$\frac{\partial T(x,y,0)}{\partial z} = -\frac{2T_0}{\pi^2} \frac{a}{a'-a} \int_0^{+\infty}\int_0^{+\infty} \left[\frac{\sin(a'\xi - b'\eta) - \sin(a\xi - b\eta)}{(a\xi - b\eta)\xi\eta} - \right.$$

$$\left. \frac{\sin(a'\xi + b'\eta) - \sin(a\xi + b\eta)}{(a\xi + b\eta)\xi\eta}\right] \sqrt{\xi^2 + \eta^2} \cos \xi x \cdot \cos \eta y\ d\xi d\eta \qquad (30)$$

Substituting the expression of this gradient in equation (29) and carrying out the necessary integration, we find the total thermal flow

$$Q = \frac{2 \cdot \lambda \, T_o \, a}{\pi^2 (a' - a)} \int_0^{+\infty} \int_0^{+\infty} \int_{-a}^{a} \int_{-b}^{b} \left[\quad \right] \sqrt{\xi^2 + \eta^2} \cos \xi x \cos \eta y \, d\xi \, d\eta \, dx dy$$

$$= \frac{8 \lambda \, T_o \, a}{\pi^2 (a' - a)} \int_0^{+\infty} \int_0^{+\infty} \left[\frac{\sin (a'\xi - b'\eta) - \sin (a\xi - b\eta)}{(a\xi - b\eta)\xi\eta} - \right.$$

$$\left. \frac{\sin (a'\xi + b'\eta) - \sin (a\xi + b\eta)}{(a\xi + b\eta)\xi\eta} \right] \frac{\sin \xi a \quad \sin \eta b}{\xi \eta} \sqrt{\xi^2 + \eta^2} \, d\xi \, d\eta \qquad (31)$$

The numerical values involved in equation (31) are calculated in like manner as those in equation (27).

Using the boundary conditions (15), physically correct solutions are obtained for the temperature field as well as the thermal flow. This will also be shown with the aid of numerical calculations, applying the equations derived under b) for a slab having the shape of a narrow strip.

FOUNDATION SLAB HAVING THE SHAPE OF A NARROW STRIP

Ground surface temperature outside the slab is 0^oC.

If $a \gg b$ (elongated buildings, terrace houses, etc.), the temperature field in the middle of the slab can be calculated by assuming the slab to be a rectangle of infinite length and of width a.

The temperature function T(x, z) shall satisfy Laplace's equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} = 0 \qquad (32)$$

and the boundary conditions

$$\begin{aligned} &a) \; T(x,0) = T_o \quad \text{when} \quad -a < x < +a \\ &b) \; T(x,0) = 0 \quad \text{"} \quad -a > x > +a \\ &c) \; T = 0 \quad \text{for} \quad z = \infty \end{aligned} \qquad (33)$$

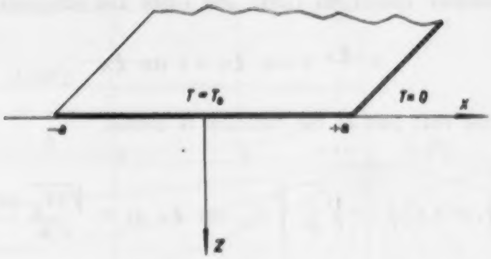

Fig. 3. Foundation slab having the shape of a narrow strip. Temperature of the slab T_0; temperature at ground surface outside the slab $0^{o}C$.

The Fourier transform

$$T^{*}(\xi, z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} T(x, z)\, e^{i\xi x}\, dx \tag{34}$$

satisfies equation (32).

According to Fourier's inversion theorem,

$$T(x, z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} T^{*}(\xi, z)\, e^{-i\xi x}\, d\xi \tag{35}$$

Equation (32) can be solved by multiplying it by $e^{i\xi x}$. Partial integration converts equation (32) into

$$\frac{\partial^2 T^{*}}{\partial z^2} = \xi^2 T^{*} \tag{36}$$

Selecting the finite branch of the exponential constituting the solution of equation (35), we find according to boundary condition (33c)

$$T^{*}(\xi, z) = C(\xi)\, e^{-\xi z}$$

By virtue of boundary conditions (33a) and (33b) and employing Euler's formula

$$e^{i\xi x} = \cos \xi x + i \sin \xi x \tag{38}$$

we have, when the real part of the function is chosen,

$$T^*(\xi,0) = C(\xi) = \sqrt{\frac{2}{\pi}} \int_0^a T_o \cos \xi x \, dx = \frac{\sqrt{2T_o}}{\sqrt{\pi}} \frac{\sin \xi a}{\xi}$$

According to equation (35),

$$T(x,z) = \frac{2T_o}{\pi} \int_0^\infty \frac{\sin \xi a}{\xi} \cos \xi x \, e^{-\xi z} \, d\xi \tag{39}$$

which is a function satisfying the boundary conditions (33).

Let us now introduce dimensionless variables by employing a as a unit of measure, and writing $\xi a = \alpha$. Equation (37) then becomes

$$T(x,z) = \frac{2}{\pi} T_o \int_0^\infty \frac{\sin \alpha}{\alpha} \cos \frac{x}{a} \alpha \, e^{-\frac{z}{a}\alpha} \, d\alpha = \frac{T_o}{\pi} \left[\text{arc tg} \frac{a+x}{z} + \text{arc tg} \frac{a-x}{z} \right] \tag{40}$$

The graphical representation of equation (40) is shown in Fig. 4.

The same result is obtained from the theory of the rectangular slab if one of its sides is allowed to increase infinitely. This is easily seen by determining the limit value which is approached when $b \rightarrow \infty$ in equation (43).

In deriving equation (39), it was assumed that the temperature decreases from T_o to zero in the manner of a step function at the edge of the slab. This is not consistent with actual conditions, the decrease of temperature in the lagging on the edge of the slab actually following a certain continuous function. If we approximate this function by means of a linear function, we find new boundary conditions:

$$\begin{array}{llr} \text{a)} \; T(x,0) = T_o & \text{when} & -a < x < a \\ \text{b)} \; \text{-"-} = T_o - \dfrac{(x-a)}{b-a} T_o & \text{"} & a < x < b \\ \text{c)} \; \text{-"-} = 0 & \text{"} & x > b \\ \text{d)} \; T \rightarrow 0 \text{ for } z \rightarrow \infty & & \end{array} \tag{41}$$

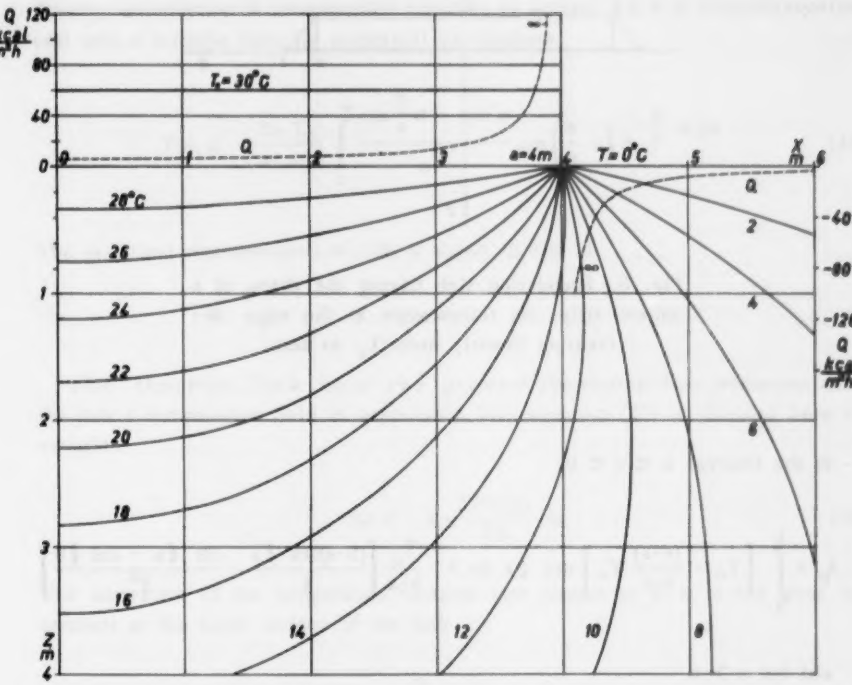

Fig. 4. Course of the isothermal lines and thermal flow from slab to ground under a foundation slab having the shape of a narrow strip according to equation (40) when $T_o = 30^oC$.

We shall determine the unknown function $C(\xi)$ in equation (37) in conformity with the new boundary conditions (41)

$$T^*(\xi,0) = C(\xi) = \sqrt{\frac{2}{\pi}} \int_0^\infty T(x,0) \cos \xi x \, dx \tag{42}$$

Integration, considering the boundary conditions (41a), (41b) and (41c), renders:
- in the interval $0 < x < a$

$$I_a = T_o \int_0^a \cos \xi x \, dx = \frac{T_o}{b-a} \, \frac{(b-a) \sin \xi a}{\xi}$$

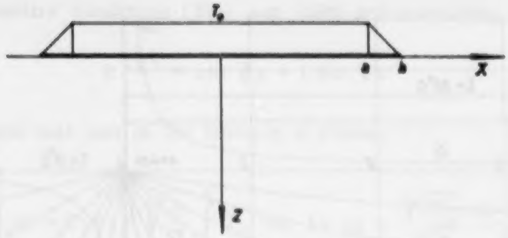

Fig. 5. Foundation slab having the shape of a narrow strip; its temperature at the edge decreasing linearly from T_o to zero.

- in the interval $a < x < b$

$$I_b = \int_a^b \left[T_o - \frac{(x-a)}{b-a} T_o \right] \cos \xi x \, dx = - \frac{T_o}{b-a} \left[\frac{(b-a)\sin \xi a}{\xi} - \frac{\cos \xi a - \cos \xi b}{\xi^2} \right]$$

- and for $x > b$

$$I_c = 0$$

$$I = I_a + I_b + I_c = \frac{T_o}{(a-b)} \frac{\cos \xi b - \cos \xi a}{\xi^2}$$

The Fourier transform for the boundary conditions on the surface will thus be

$$T^*(\xi, 0) = C(\xi) = \sqrt{\frac{2}{\pi}} \frac{T_o}{(a-b)} \frac{\cos \xi b - \cos \xi a}{\xi^2} \tag{43}$$

According to equations (35), (37), (38) and (43), the temperature function satisfying boundary conditions (41) will be

$$T(x,z) = \frac{2}{\pi} \frac{T_o}{(a-b)} \int_0^\infty \frac{\cos \xi b - \cos \xi a}{\xi^2} \cos \xi x \, e^{-\xi z} \, d\xi \tag{44}$$

Finally, introduction of dimensionless variables by writing $\xi a = \alpha$ converts equation (44) into a suitable form for numerical calculations

$$T(x,z) = \frac{2a\ T_0}{\pi(a-b)} \int_0^\infty \frac{\cos\frac{b}{a}\alpha - \cos\alpha}{\alpha^2} \cos\left(\frac{\alpha}{a}x\right) e^{-\frac{\alpha}{a}z} d\alpha \tag{45}$$

The graphical representation of (45) is shown in Fig. 6.

The thermal flow into the ground. The thermal flow production under the slab a temperature field in accordance with equation (43) is obtained from the equation

$$dQ = -\lambda \frac{\partial T(x,0)}{\partial z} dx \tag{46}$$

The derivative of the temperature function with respect to z at z = 0 gives the gradient at the lower surface of the slab

$$\frac{\partial T(x,0)}{\partial z} = -\frac{2T_0}{\pi(a-b)} \int_0^\infty \frac{\cos\xi b - \cos\xi a}{\xi} \cos\xi x\ d\xi$$

The total thermal flow is

$$Q = \frac{4\lambda T_0}{\pi(a-b)} \int_0^\infty \int_0^\xi \frac{\cos\xi b - \cos\xi a}{\xi} \cos\xi x\ d\xi\ dx$$

$$= \frac{4\lambda T_0}{\pi(a-b)} \int_0^\infty \frac{(\cos\xi b - \cos\xi a)\sin\xi a}{\xi^2} d\xi \tag{47}$$

The temperature field and thermal flow for a slab having the shape of a narrow strip, calculated by equations (45) and (47), mm are shown in Fig. 6. They are well consistent with the results of measurements in homogeneous ground when the temperature field and thermal flow inherent in the ground are superimposed on those shown by Fig. 6.

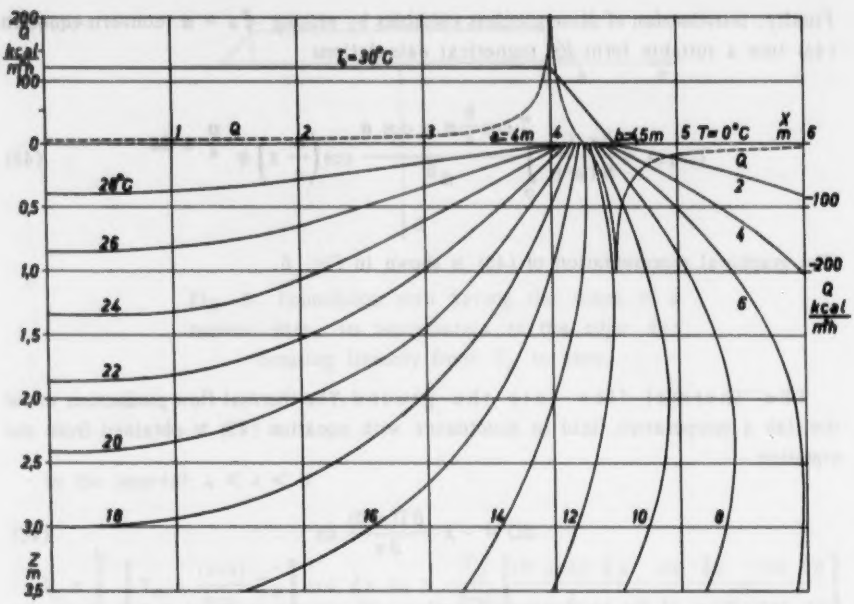

Fig. 6. Course of the isothermal lines and thermal flow from slab to ground under a foundation slab having the shape of a narrow strip according to equation (45) when $T_o = 30^oC$. The downward heat flow is positive.

The integral

$$Q = \frac{4T_o\lambda}{\pi} \int_0^\infty \int_0^a \sin\ \xi a\ \cos\ \xi x\ dx\ d\xi = \frac{4T_o\lambda}{\pi} \int_0^\infty \frac{\sin^2\ \xi a}{\xi}\ d\xi \qquad (48)$$

representing the total thermal flow calculated from the temperature function (39) derived on the basis of the boundary conditions (33) is not finite. Integration with respect to x from 0 to x gives a formula for the thermal flow in a zone from the centerline to a line at the distance x

$$Q = \frac{T_o\lambda}{\pi} \log\left(\frac{x+a}{x-a}\right) \qquad (49)$$

This equation can only be used in the central part of the slab.

The temperature function $T(x, z)$ defined by equation (44), p. 24 approaches as its limit the temperature function in equation (39), p. 22 when b approaches a. In order to show this, we write

$$\lim_{b \to a} \frac{\cos \xi b - \cos \xi a}{b - a} = 2 \lim_{b \to a} \sin \frac{\xi (b + a)}{2} \lim_{b \to a} \frac{\sin \frac{\xi (b - a)}{2}}{b - a}$$

$$= 2 \sin \xi a . \frac{\xi}{2} = \xi \sin \xi a$$

by which operation the equation (44) is transformed into

$$T(x, z) = \frac{2T_0}{\pi} \int_0^\infty \frac{\sin \xi a}{\xi} \cos \xi x \, e^{-\xi z} \, d\xi$$

which is the same as equation (39). The consideration shows that the calculations have been correctly performed.

CIRCULAR FOUNDATION SLAB

In the following, the stationary-state temperature field and thermal flow under a circular foundation slab shall be considered, choosing the boundary conditions in three different ways.

The surroundings of the slab are completely insulated

We introduce the cylindrical coordinates r, φ, z. The temperature is independent of φ and Laplace's expression has the form

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0, \tag{50}$$

This is a Besselian differential equation of the order zero.

The temperature function $T(r, z)$ shall satisfy the equation (50) and the following boundary conditions:

$$\begin{array}{lll} \text{a)}\ T(r,0) = T_o & \text{when} & 0 < r < R \\ \text{b)}\ \dfrac{\partial T(r,0)}{\partial z} = 0 & \text{"} & r > R \\ \text{c)}\ \dfrac{\partial T(r,z)}{\partial r} = 0 & \text{"} & r = 0 \\ \text{d)}\ T(r,z) \rightarrow 0 & \begin{array}{l}\text{"}\\ \text{"}\end{array} & \left\{\begin{array}{l} z \rightarrow \infty \\ r \rightarrow \infty \end{array}\right. \end{array} \qquad (51)$$

The Hankel transform of the order zero of $T(r,z)$

$$T^*(\xi, z) = \int_0^\infty r\ T(r,z)\ J_o(\xi r)dr, \quad z > 0 \qquad (52)$$

satisfies equation (50).

We subject equation (50) to Hankel's transformation:

$$\frac{\partial^2 T^*(\xi,z)}{\partial z^2} = \int_0^\infty r\ \frac{\partial^2 T(r,z)}{\partial z^2}\ J_o(\xi r)dr$$

$$= -\int_0^\infty r \left[\frac{\partial^2 T(r,z)}{\partial r^2} + \frac{1}{r}\frac{\partial T(r,z)}{\partial r}\right] J_o(\xi r)dr = \xi \int_0^\infty r\ \frac{\partial T}{\partial r}\ J_o'(\xi r)dr$$

$$= -\xi^2 \int_0^\infty r\ T(r,z) \left[J_o''(\xi r) + \frac{1}{\xi r} J_o'(\xi r)\right] dr = \xi^2 T^*(\xi, z)$$

The Besselian differential equation (50) is converted into a linear, homogeneous differential equation of the second order,

$$\frac{\partial^2 T^*(\xi, z)}{\partial z^2} = \xi^2\ T^*(\xi, z) \qquad (53)$$

The Hankel transform of the temperature function thus satisfies the differential equation (53).

The general solution of equation (53) is

$$T^*(\xi, z) = A(\xi) e^{-\xi z} + B(\xi) e^{+\xi z} \qquad (54)$$

In compliance with boundary condition (51c), we choose the finite branch of the function, which implies that $B(\xi) = 0$

$$T^*(\xi, z) = A(\xi)\, e^{-\xi z} \tag{55}$$

The unknown function $A(\xi)$ in this equation will be determined by the boundary conditions. From equations (52) and (55), we obtain

$$\int_0^\infty r\, T(r, z)\, J_o(\xi r)dr = A(\xi)\, e^{-\xi z}\,, \qquad z > 0$$

According to Hankel's inversion theorem, we have further

$$T(r, z) = \int_0^\infty \xi\, A(\xi)\, e^{-\xi z}\, J_o(\xi r)d\xi \tag{56}$$

Differentiating equation (56) with respect to z, we find

$$\frac{\partial T(r, z)}{\partial z} = -\int_0^\infty \xi^2\, A(\xi)\, e^{-\xi z}\, J_o(\xi r)d\xi \tag{57}$$

Applying the boundary conditions (51a) and (51b) to equations (56) and (57), we obtain the dual equations

$$T(r, 0) = \int_0^\infty \xi\, A(\xi)\, J_o(\xi r)d\xi = T_o \qquad 0 < r < R$$

(58)

$$\frac{\partial T(r, 0)}{\partial z} = -\int_0^\infty \xi^2\, A(\xi)\, J_o(\xi r)d\xi = 0 \qquad r > R$$

Choosing the radius R of the circular slab as a unit of measure and writing

$$\frac{r}{R} = \varrho, \quad \xi = \frac{u}{R} \qquad \begin{array}{l} 0 < r < R \\ 0 < \varrho < 1 \end{array}$$

we have

$$\int_0^\infty \xi A(\xi)\, J_o(\xi r)d\xi = \int_0^\infty \frac{1}{R^2}\, u\, A\left(\frac{u}{R}\right) J_o(\varrho u)du = T_o$$

$$\int_0^\infty \xi^2\, A(\xi)\, J_o(\xi r)d\xi = \int_0^\infty \frac{1}{R^3}\, u^2\, A\left(\frac{u}{R}\right) J_o(\varrho u)du = 0$$

Let be $u^2\, A\left(\frac{u}{R}\right) = f(u)$; then $u\, A\left(\frac{u}{R}\right) = u^{-1}\, f(u)$, and the dual integral equations (57) are converted into

$$\int_0^\infty u^{-1}\, f(u)\, J_o(u\varrho)du = R^2 T_o \qquad 0 < \varrho < 1$$

$$\int_0^\infty f(u)\, J_o(u\varrho) = 0 \qquad \varrho > 1 \tag{59}$$

The function f(u) is determined with the aid of these two equations.

The equations just obtained constitute a particular case of the more generally valid dual integral equations

$$\int_0^\infty u^\alpha\, f(u)\, J_\nu\, (u\varrho)du = g(\varrho) \qquad 0 < \varrho < 1$$

$$\int_0^\infty f(u)\, J_\nu\, (u\varrho)du = 0 \qquad 1 < \varrho \tag{60}$$

where $g(\varrho)$ is a known function and f(u) is being sought.

The dual integral equations were first solved by Titmarsch [11] with the aid of Mellin's transformation. His solution is only valid if $\alpha > 0$. The equations have also been solved by I.W. Busbridge, his solution being valid when $\alpha > -2$ and $-\nu - 1 < \left(\alpha - \frac{1}{2}\right) < \nu + 1$, provided that g(u) is integrable in the interval 0 to 1 [12].

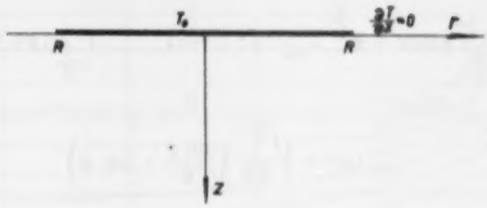

Fig. 7. Circular foundation slab. Constant temperature of the slab = T_0; surroundings of the slab completely insulated, $\frac{\partial T}{\partial z} = 0$.

Busbridge's solution is

$$f(\varrho) = \frac{2^{-\frac{1}{2}\alpha}\,\varrho^{-\alpha}}{\Gamma\left(1+\frac{1}{2}\alpha\right)} \left[\varrho^{\frac{1}{2}\alpha+1} J_{\nu+\frac{1}{2}\alpha}(\varrho) \int_0^1 u^{\nu+1}(1-u^2)^{\frac{1}{2}\alpha} g(u)du \; + \right.$$

$$\left. \int_0^1 v^{\nu+1}(1-v^2)^{\frac{1}{2}\alpha} dv \int_0^1 g(uv)\,(\varrho u)^{2+\frac{1}{2}\alpha} J_{\nu+1+\frac{1}{2}\alpha}(\varrho u)du \right]$$

In the case that $\alpha = -1$ and $\nu = 0$, and $g(u) = R^2T_0$, the unknown function in the equations (59) is

$$f(\varrho) = \frac{R^2T_0 2^{\frac{1}{2}}\varrho}{\Gamma\left(\frac{1}{2}\right)} \left[\varrho^{\frac{1}{2}} J_{-\frac{1}{2}}(\varrho) \int_0^1 u(1-u^2)^{\frac{1}{2}} du \; + \right.$$

$$\left. \int_0^1 v(1-v^2)^{-\frac{1}{2}} dv \int_0^1 (\varrho u)^{3/2} J_{\frac{1}{2}}(\varrho u)du \right] \qquad (61)$$

We substitute in equation (61)

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi} \qquad \int_0^1 \frac{u}{\sqrt{1-u^2}}\,du = 1 \qquad \int_0^1 \frac{v}{\sqrt{1-v^2}} = 1$$

$$\int_0^1 (\varrho u)^{3/2} J_{\frac{1}{2}}(\varrho u)du = \varrho^{\frac{1}{2}} J_{3/2}(\varrho) \qquad J_{-\frac{1}{2}}(\varrho) = \sqrt{\frac{2}{\pi\varrho}} \cos \varrho$$

$$J_{3/2}(\varrho) = \sqrt{\frac{2}{\pi\varrho}} \left(\frac{\sin \varrho}{\varrho} - \cos \varrho\right)$$

by which this equation is converted to

$$f(\varrho) = \frac{R^2 T_0 \sqrt{2\varrho}}{\pi} \left[\varrho^{\frac{1}{2}} \sqrt{\frac{2}{\pi\varrho}} \cos \varrho + \varrho^{\frac{1}{2}} \sqrt{\frac{2}{\pi\varrho}} \left(\frac{\sin \varrho}{\varrho} - \cos \varrho\right)\right] = \frac{2R^2 T_0}{\pi} \sin \varrho \tag{62}$$

The unknown function in the equations (59) is thus

$$f(u) = \frac{2R^2 T_0}{\pi} \sin u \tag{63}$$

and the unknown function $A(\xi)$ is obtained by re-introducing the initial variables $u = R\xi$

$$A(\xi) = \frac{f(R\xi)}{R^2 \xi^2} = \frac{2T_0}{\pi} \frac{\sin R\xi}{\xi^2} \tag{64}$$

The final form of the temperature function (56) will then be

$$T(r,z) = \frac{2T_0}{\pi} \int_0^\infty \frac{\sin R\xi}{\xi} e^{-\xi z} J_0(\xi r)d\xi \tag{65}$$

On the boundary surface $z = 0$, the integral occurring in the solution becomes $\frac{\pi}{2}$ for $r = R$ and $\arcsin \frac{R}{r}$ for $r > R$ [13].

The temperature function $T(r,0)$ equals T_0 for $r = R$ and $T(r,0) = \frac{2}{\pi} T_0 \arcsin \frac{R}{r}$ for $r > R$. When $r \to \infty$, $T(r,0)$ approaches the limiting value 0. The stipulated boundary conditions (51a) and (51c) are thus satisfied.

There is

$$\frac{\partial T(r,0)}{\partial z} = \frac{2T_0}{\pi} \int_0^\infty \sin R\xi \, J_0(\xi r)d\xi = 0 \qquad \text{when } r > R$$

and equation (65) thus satisfies also the boundary condition (51b). In the solution outlined here, the unknown function was sought by applying a systematic procedure.

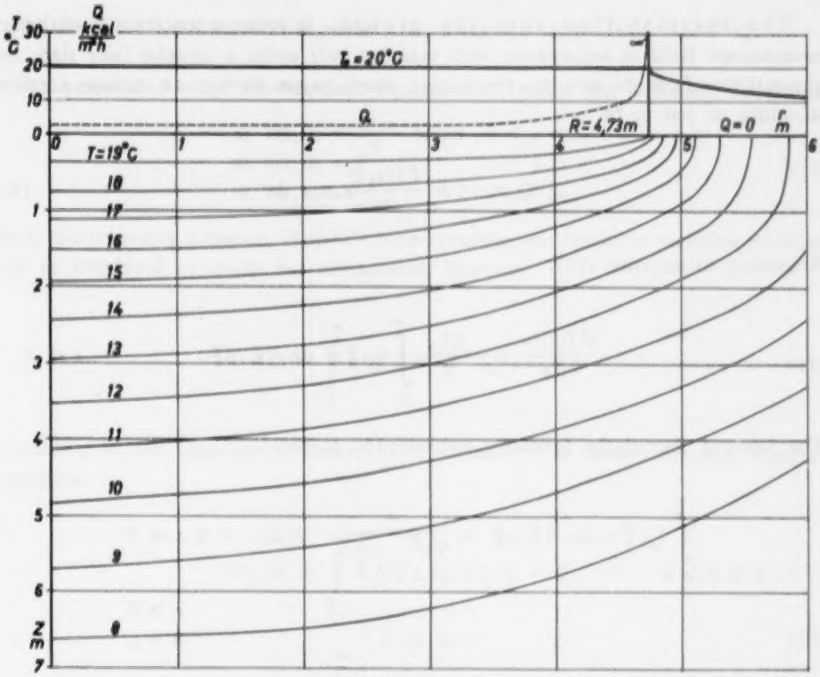

Fig. 8. Course of the isothermal lines and thermal flow from slab to ground under a circular foundation slab according to equation (66) when $T_o = 20^oC$. The downward heat flow is positive.

It can also be found by trial [14]. For the purpose of numerical calculations, we write $R\xi = \alpha$, i.e., we choose the radius of the circle, R, as a unit of length. The integral will then be

$$T(r,z) = \frac{2T_o}{\pi} \int_0^\infty \frac{\sin\alpha}{\alpha} e^{-\frac{z}{r}\alpha} J_o\left(\frac{r}{R}\alpha\right) d\alpha \qquad z > 0 \qquad (66)$$

The temperature field defined by equation (66) is shown in Fig. 8. When the ground surface around a circular slab is completely insulated (boundary condition 2), the isothermal lines are perpendicular to the ground surface outside the slab. This condition is approximated in actual practice when the ground surface is covered by a thick layer of snow.

The thermal flow into the ground. In order to maintain a stationary temperature field in accordance with equation (65) under a circular floor slab, the thermal flow from the slab into the ground which passes through an annular element of width dr has to be

$$dQ = -2\lambda \frac{\partial T(r,0)}{\partial z} \pi r \, dr \tag{67}$$

According to equation (65),

$$\frac{\partial T(r,0)}{\partial z} = -\frac{2T_0}{\pi} \int_0^\infty J_0(\xi r) \sin R\xi \, d\xi \qquad r < R$$

The integral obtains the following values [13; p. 405]

$$\int_0^\infty J_0(\xi r) \sin R\xi \, d\xi = \frac{1}{\sqrt{R^2 - r^2}} \quad \text{when} \quad 0 < r < R$$

$$= 0 \qquad \text{"} \qquad r > R$$

$$= \infty \text{ or } 0 \qquad \text{"} \qquad R = r$$

Equation (67) is thus converted into

$$dQ = +\frac{4T_0\lambda}{\pi} \frac{\pi r \, dr}{\sqrt{R^2 - r^2}} \tag{68}$$

The total heat quantity flowing from the slab into the ground is obtained by integration:

$$Q = +4T_0\lambda \int_0^R \frac{r \, dr}{\sqrt{R^2 - r^2}} = 4T_0\lambda R \tag{69}$$

It can be seen from this equation that the total thermal flow from the slab into the ground is directly proportional to the thermal conductivity of the soil, λ, and to the radius of the slab, R. As the floor area of a building increases with the square of this radius, it can be inferred that large buildings are more advantageous than small buildings from the viewpoint of heat losses.

Equation (68) reveals that the losses increase rapidly towards the edge (when $r \to R$) and become infinite on the boundary line, while the total thermal flow itself remains finite (Fig. 8).

Ground surface temperature outside the slab is $0^{o}C$.

We proceed to solve the Besselian equation (50) under the boundary conditions

$$\begin{aligned} &a)\ T(r,0) = T_o \quad \text{when} \quad 0 < r < R \\ &b)\ T(r,0) = 0 \quad \text{"} \quad r > R \\ &c)\ T(r,z) \rightarrow 0 \quad \text{"} \quad z \rightarrow \infty \end{aligned} \tag{70}$$

As in the preceding instance, Hankel's transformation and Hankel's inversion theorem can be employed to obtain the temperature function

$$T(r,z) = \int_0^\infty \xi A(\xi) e^{-\xi z} J_o(\xi r)\, d\xi \tag{71}$$

According to the boundary conditions (70), equation (71) renders the dual integral equations

$$\begin{aligned} T(r,0) &= \int_0^\infty \xi A(\xi) J_o(\xi r) d\xi = T_o \qquad 0 < r < R \\ T(r,0) &= \int_0^\infty \xi A(\xi r) J_o(\xi r) d\xi = 0 \qquad r > R \end{aligned} \tag{72}$$

As a new unit, we introduce the radius of the circular slab, writing

$$\frac{r}{R} = \varrho \quad \text{and} \quad \frac{u}{R} = \xi \qquad \begin{matrix} 0 < r < R \\ 0 < \varrho < 1 \end{matrix} \qquad \begin{matrix} \text{when } \xi \text{ increases from } 0 \text{ to } \rightarrow \infty \\ \text{then } u \text{ increases from } 0 \text{ to } \rightarrow \infty \end{matrix}$$

$$\xi r = \varrho u \qquad uA\left(\frac{u}{R}\right) = f(u)$$

The dual integral equations (72) now become

$$\begin{aligned} &\int_0^\infty f(u) J_o(\varrho u) du = R^2 T_o \qquad 0 < \varrho < 1 \\ &\int_0^\infty f(u) J_o(\varrho u) du = 0 \qquad \varrho > 1 \end{aligned} \tag{73}$$

These equations represent another particular case of the dual integral equations (60) (for $\alpha = 0$, $\nu = 0$ and $g(u) \equiv R^2T_o$).

In this case, too, the solution of equations (72) is obtained from Busbridge's solution by substituting $\alpha = 0$, $\nu = 0$ and $g(\varrho) = R^2T_o$, for $g(\varrho)$ is integrable in the interval 0 to 1 and regular for $y \to \infty$. Moreover, the conditions $\alpha > -2$ and $-\nu - 1 < \left(\alpha - \frac{1}{2}\right) < \nu + 1$ are satisfied.

$$f(\varrho) = \varrho J_o(\varrho) \int_0^1 R^2T_o u\, du + \int_0^1 v\, dv \int_0^1 R^2T_o (\varrho u)^2 J_1(\varrho u) du =$$

$$= \frac{1}{2} R^2T_o \left[\varrho J_o(\varrho) + \int_0^1 (\varrho u)^2 J_1(\varrho u) du \right]$$

We introduce a new variable, writing

$$\varrho u = t, \qquad u = \frac{t}{\varrho}, \qquad du = \frac{1}{\varrho} dt$$

and carry out the integration

$$\int_0^1 (\varrho u)^2 J_1(\varrho u) du = \frac{1}{\varrho} \int_0^{\varrho} t^2 J_1(t) dt = \varrho J_2(\varrho)$$

$$\int_0^{\varrho} t^2 J_1(t) dt = \varrho^2 J_2(\varrho)$$

Hence

$$f(\varrho) = \frac{1}{2} R^2T_o \left[\varrho J_o(\varrho) + \varrho J_2(\varrho) \right] = \frac{1}{2} R^2T_o\, \varrho \left[J_o(\varrho) + J_2(\varrho) \right] =$$

$$\frac{1}{2} R^2T_o\, \varrho \frac{2}{\varrho} J_1(\varrho) = R^2T_o J_1(\varrho)$$

$$f(\varrho) = R^2T_o J_1(\varrho) \tag{74}$$

Re-introducing the initial variables

$$f(u) = R^2T_o J_1(u)$$

$$A(\xi) = RT_o \frac{J_1(R\xi)}{\xi}$$

we find the final form of the temperature function (71)

$$T(r,z) = RT_o \int_0^\infty J_o(r\xi)\, J_1(R\xi) e^{-\xi z}\, d\xi \tag{75}$$

At ground surface (for $z = 0$) this temperature function has the values

$$T(r,0) = RT_o \int_0^\infty J_o(r\xi)\, J_1(R\xi)\, d\xi = \begin{cases} 0 & \text{when } r > R \\ \frac{T_o}{2} & \text{"} \quad r = R \\ T_o & \text{"} \quad r < R \end{cases}$$

With a view to easier numerical calculations, we choose the radius of the slab for the unit of measurement and write $R\xi = \alpha$; $Rd\xi = d\alpha$; $\xi = \frac{\alpha}{R}$. We have then

$$T(r,z) = T_o \int_0^\infty J_o\left(\alpha \frac{r}{R}\right) J_1(\alpha)\, e^{-\frac{z}{R}\alpha}\, d\alpha \tag{76}$$

The isothermal lines of the temperature field consistent with the boundary conditions (70), calculated by equation (76), have been shown in Fig. 9.

The temperature gradient calculated by equation (71) for $z = 0$

$$\frac{\partial T(r,0)}{\partial z} = -RT_o \int_0^\infty J_o(r\xi)\, J_1(R\xi)\xi\, d\xi$$

and the total thermal flow

$$Q = 2\pi\lambda R^2 T_o \int_0^\infty J_1^2\, (R\xi) d\xi$$

become infinite. No physically correct result is thus obtained under the boundary conditions (70).

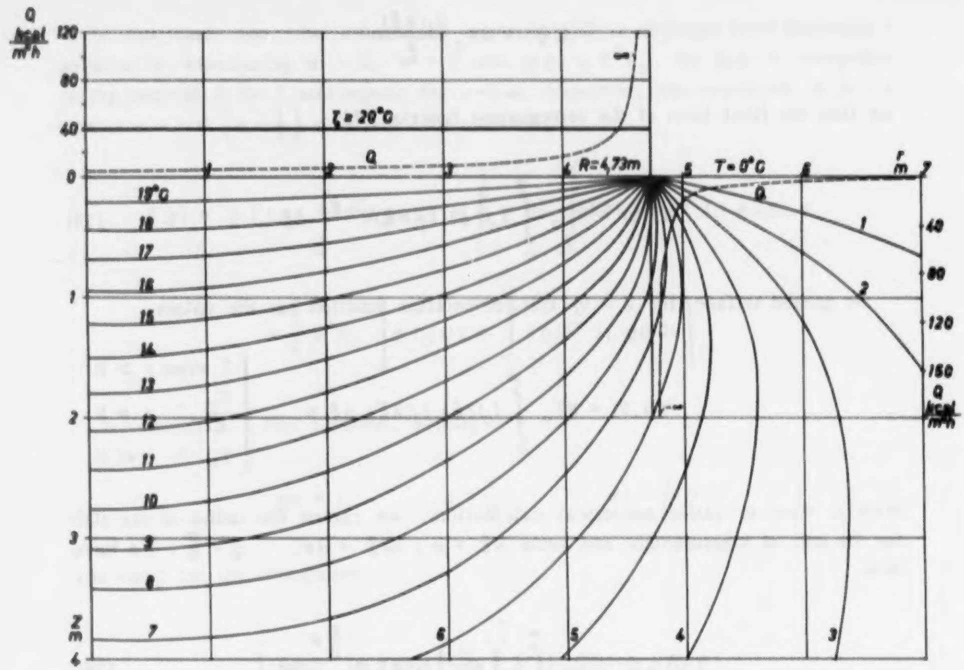

Fig. 9. Course of the isothermal lines and thermal flow from slab to ground under a circular foundation slab according to equation (76) when $T_o = 20^oC$. The downward heat flow is positive.

Constant thermal flow from the slab, and completely insulated surroundings

Let a constant thermal flow Q_o be conveyed into the ground through the area under a circular slab, assuming that the thermal flow at ground surface outside the slab is zero.

Owing to symmetry of the temperature field, Laplace's expression in polar coordinates is not dependent on φ, and it has the form

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{77}$$

which is a Besselian differential equation of the order zero. The temperature function T(r, z) shall satisfy equation (77) and the boundary conditions

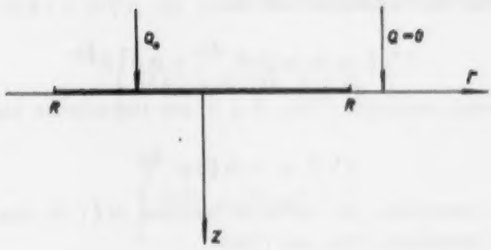

Fig. 10. Circular foundation slab; the thermal flow Q_o entering the ground from the slab being known. Completely insulated surroundings of the slab.

$$\begin{aligned}
&\text{a)}\quad \frac{\partial T(r,0)}{\partial z} = -\frac{1}{\lambda} Q_o && \text{when } 0 \leq r < R\\
&\text{b)}\quad \frac{\partial T(r,0)}{\partial z} = 0 && \text{"} \quad r > R\\
&\text{c)}\quad \frac{\partial T(0,z)}{\partial r} = 0 && (78)\\
&\text{d)}\quad T(r,z) \rightarrow 0 \begin{cases} r \rightarrow \infty \\ z \rightarrow \infty \end{cases}
\end{aligned}$$

Hankel's transform of the order zero of the temperature function $T(r,z)$ for the derivatives,

$$T^*(\xi, z) = \int_0^\infty r\ T(r,z)\ J_o(\xi r)dr, \qquad z > 0 \tag{79}$$

satisfies equation (77).

We subject equation (77) to Hankel's transformation

$$\frac{\partial^2 T^*(\xi,z)}{\partial z^2} = \int_0^\infty r\ \frac{\partial^2 T(r,z)}{\partial z^2} J_o(\xi r)dr = -\int_0^\infty r \left[\frac{\partial^2 T(r,z)}{\partial r^2} + \frac{1}{r}\frac{\partial T(r,z)}{\partial r}\right] J_o(\xi r)\, dr$$

and expand the integral by partial integration. Taking into account the boundary conditions (78c) and (78d), the Besselian equation (77) is then found to be converted into a linear, homogeneous equation of the second order

$$\frac{\partial^2 T^*(\xi,z)}{\partial z^2} = \xi^2 T^*(\xi, z) \tag{80}$$

which is satisfied by Hankel's transform (79) of the temperature function.

The general solution of equation (80) is

$$T^*(\xi, z) = A(\xi)e^{-\xi z} + B(\xi)e^{\xi z} \tag{81}$$

By virtue of boundary condition (78c), $B \equiv 0$ and the solution has the form

$$T^*(\xi, z) = A(\xi)e^{-\xi z} \tag{82}$$

We shall then determine the unknown function $A(\xi)$ in equation (82) on the basis of boundary conditions (78a) and (78b).

According to Hankel's inversion theorem, we obtain from equation (79) the temperature function

$$T(r, z) = \int_0^\infty \xi T^*(\xi, z)\, J_0(\xi r) d\xi \qquad z > 0 \tag{83}$$

Equation (82) can be used to convert the temperature function (83) into

$$T(r, z) = \int_0^\infty \xi\, A(\xi)e^{-\xi z}\, J_0(\xi r) d\xi \tag{84}$$

and we find by derivation

$$\frac{\partial T(r, z)}{\partial z} = -\int_0^\infty \xi^2\, A(\xi)e^{-\xi z}\, J_0(\xi r) d\xi$$

and according to boundary conditions (78a) and (78b) in the plane $z = 0$

$$\frac{\partial T(r, 0)}{\partial z} = -\int_0^\infty \xi^2\, A(\xi)\, J_0(\xi r) d\xi = -\frac{1}{\lambda} Q_0 \qquad 0 < r < R$$

$$\frac{\partial T(r, 0)}{\partial z} = \int_0^\infty \xi^2\, A(\xi)\, J_0(\xi r) d\xi = 0$$

The unknown function $A(\xi)$ is determined from the dual integral equations

$$\int_0^\infty \xi^2\, A(\xi)\, J_0(\xi r) d\xi = \frac{1}{\lambda} Q_0 \qquad 0 < r < R$$

$$\int_0^\infty \xi^2\, A(\xi)\, J_0(\xi r) d\xi = 0 \qquad r > R \tag{85}$$

We write $\xi^2 A(\xi) = G(\xi)$, by which the equations (85) are converted into

$$\int_0^\infty G(\xi)\, J_0(\xi r)d\xi = \frac{1}{\lambda} Q_0 \qquad 0 < r < R$$

$$\int_0^\infty G(\xi)\, J_0(\xi r)d\xi = 0 \qquad R < r$$

This is a particular case of the dual equations (60), p. 30, for $\alpha = 0$ and $\nu = 0$

$$\frac{r}{R} = \varrho, \quad \frac{u}{R} = \xi, \qquad \begin{matrix} \xi = 0 \\ u = 0 \end{matrix} \quad \begin{cases} \xi = \infty \\ u = \infty \end{cases}$$

$$\xi r = \varrho u$$

$$\int_0^\infty G\left(\frac{u}{R}\right) J_0(\varrho u)du = R\frac{Q_0}{\lambda}$$

$$\int_0^\infty G\left(\frac{u}{R}\right) J_0(\varrho u)du = 0$$

We write $G\left(\frac{u}{R}\right) = f(u)$;

$$\int_0^\infty f(u)\, J_0(\varrho u)du = R\frac{Q_0}{\lambda} \qquad 0 < \varrho < 1$$

(86)

$$\int_0^\infty f(u)\, J_0(\varrho u)du = 0 \qquad \varrho > 1$$

The solution of the dual equations is found to be (cf. p. 36)

$$f(\varrho) = R\frac{Q_0}{\lambda} J_1(\varrho)$$

$$f(u) = \frac{RQ_0}{\lambda} J_1(u)$$

$$G\left(\frac{u}{R}\right) = f(u)$$

$$A(\xi) = \frac{1}{\xi^2} G(\xi) = \frac{RQ_0}{\lambda}\frac{J_1(u)}{\xi^2}$$

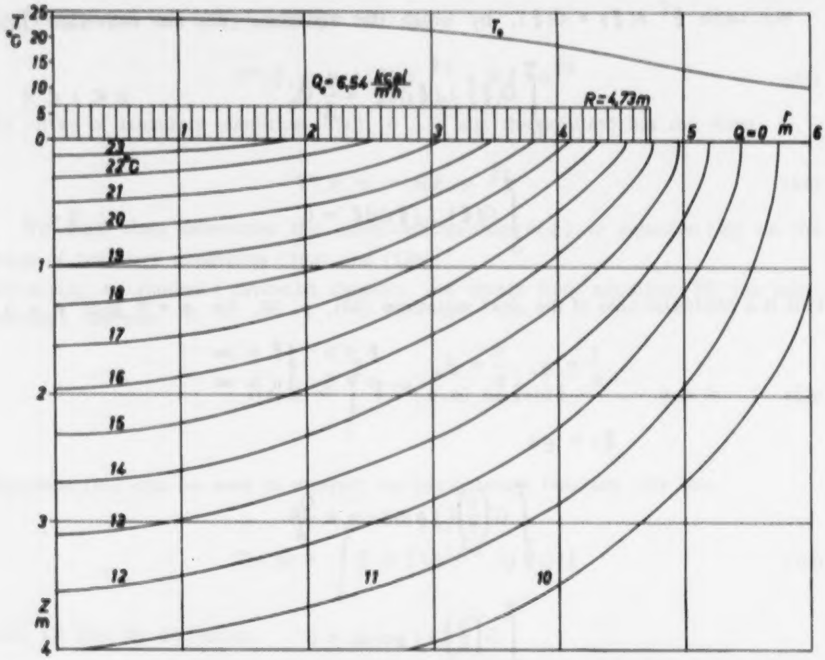

Fig. 11. Course of the isothermal lines under a circular foundation slab, and surface temperature as a function of the radius, according to equation (87), when the thermal flow entering the ground from the slab is $Q_0 = 6,54$ kcal/m^{2}h.

Substituting the expression for $A(\xi)$ in equation (84), the temperature function is found to be (Fig. 11)

$$T(r,z) = \frac{RQ_0}{\lambda} \int_0^\infty \frac{J_0(\xi r)}{\xi} J_1(\xi R) e^{-\xi z} \, d\xi \tag{87}$$

which satisfies the stipulated boundary conditions. (Cf. Weber-Schaftheitl's integral [13; p. 398].

NON-STATIONARY FIELD

SLAB HAVING THE SHAPE OF A NARROW STRIP; ITS SURFACE TEMPERATURE VARYING AS A FUNCTION OF TIME

In the cases dealt with in the preceding chapter, the temperature of the heated slab was assumed to be constant and the temperature field in a stationary state was considered. However, the temperature of a foundation slab varies during the heating period according to the heat requirements of the building, depending on weather conditions, and it follows the temperature changes of ambient air. The thermal flow from the slab into the ground, Q (in kcal per m^2 and hour) is also a function of time, which is known on the basis of measurements

$$Q(y,t) = -\lambda \frac{\partial T}{\partial z} \qquad (88)$$

Outside the slab, along the wall of the building, the snow that has fallen during the winter, together with snow coming down from the roof, forms a thick layer of high thermal insulating capacity. The temperature at ground surface is therefore close to $0^{o}C$. In the mathematical treatment it will be assumed to equal zero.

The temperature field produced by the heated slab in the ground is a periodic function of time $T(y,z,t)$, upon which the periodic temperature field of the ground itself, $T(z,t)$, is superimposed.

The function $T_2(y,z,t)$ representing the temperature field produced by the slab satisfies the differential equation

$$\frac{\partial T}{\partial t} = a\left[\frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right] \qquad (89)$$

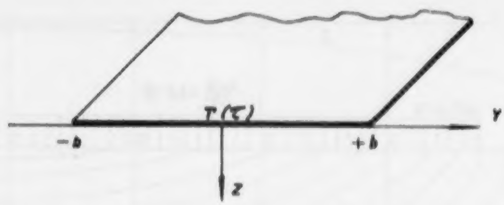

Fig. 12. Foundation slab having the shape of a narrow strip; its temperature being a given function of time, T(t).

if the ground is assumed to consist of a homogeneous, isotropic substance having constant temperature conductivity a. We solve this equation under the boundary conditions

$$T(y, z, 0) = 0 \tag{90}$$

and

$$\begin{aligned}
&\text{a)}\ T(y,0,t) = T(y,t) = T(t) \quad \text{when } (-b < y < b)\\
&\qquad\ \text{-" -} \quad = 0 \qquad\qquad\qquad \text{"} \qquad\quad /y/ > b\\
&\text{b)}\ \frac{\partial}{\partial y} T(0,z,t) = 0\\
&\text{c)}\ T(y,z,t) \rightarrow 0 \quad \text{when } y \text{ and } z \rightarrow \infty
\end{aligned} \tag{91}$$

In order to find the solution of equation (89), it is converted into a linear, homogeneous differential equation by means of Fourier's sine and cosine transformations

$$T^*(\eta, \zeta, t) = \frac{2}{\pi} \int_0^\infty \int_0^\infty T(y,z,t) \cos \eta y \sin \zeta z \, dy \, dz \tag{92}$$

To this purpose, both members of equation (89) are multiplied by $\frac{2}{\pi} \cos \eta y \sin \zeta z$ and integrated

$$\frac{2}{\pi} \int_0^\infty \int_0^\infty \frac{\partial T}{\partial t} \cos \eta y \sin \zeta z \, dydz = \frac{2a}{\pi} \int_0^\infty \int_0^\infty \left(\frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) \cos \eta y \sin \zeta z \, dydz$$

$$\frac{2}{\pi} \int_0^\infty \int_0^\infty \frac{\partial T}{\partial t} \cos \eta y \sin \zeta z \, dydz = \frac{\partial T^*}{\partial t}$$

$$\frac{2a}{\pi}\int_0^\infty\int_0^\infty \frac{\partial^2 T}{\partial z^2}\cos\eta y\sin\zeta z\,dy\,dz =$$

$$\frac{a\sqrt{2}}{\sqrt{\pi}}\sqrt{\frac{2}{\pi}}\int_0^\infty \zeta T(y,0,t)\cos\eta y dy - \frac{2a}{\pi}\zeta^2\int_0^\infty\int_0^\infty T(y,z,t)\cos\eta y\sin\zeta z\,dydz =$$

$$\frac{a\sqrt{2}}{\sqrt{\pi}}\zeta T_c^*(\eta,t) - a\zeta^2 T^*(\eta,\zeta,t)$$

Second integral: Taking into account boundary conditions (91b) and (91c), we find

$$\frac{2a}{\pi}\int_0^\infty\int_0^\infty \frac{\partial^2 T}{\partial y^2}\cos\eta y\sin\zeta z\,dydz = -a\eta^2 T^*(\eta,\zeta,t)$$

Equation (89) is thus converted into a linear, homogeneous differential equation of the first order

$$\frac{\partial T^*}{\partial t} + a(\eta^2+\zeta^2)T^*(\eta,\zeta,t) - \frac{a\sqrt{2}}{\sqrt{\pi}}\zeta T_c^*(\eta,t) = 0 \tag{93}$$

Its solution can be obtained, e.g., by the method of variable constants [9; p. 49]

$$T^*(\eta,\zeta,t) = e^{-a(\eta^2+\zeta^2)\int_0^t dt}\left[T^*(\eta,\zeta,0) + \frac{a\sqrt{2}}{\sqrt{\pi}}\int_0^t \zeta e^{a(\eta^2+\zeta^2)\int_0^\tau d\tau}\,T^*(\eta,\tau)d\tau\right]$$

$$= T^*(\eta,\zeta,0)e^{-at(\eta^2+\zeta^2)} + \frac{a\sqrt{2}}{\sqrt{\pi}}\int_0^t \zeta T_c^*(\eta,\tau)e^{-a(t-\tau)(\eta^2+\zeta^2)}\,d\tau \tag{94}$$

According to boundary equation (90),

$$T^*(\eta,\zeta,0) = \frac{2}{\pi}\int_0^\infty\int_0^\infty T(y,z,0)\cos\eta y\cos\zeta z\,dydz = 0$$

The solution (93) of the linear equation has therefore the form

$$T^*(\eta,\zeta) = \frac{a\sqrt{2}}{\sqrt{\pi}}\int_0^t \zeta T_c^*(\eta,\tau)\,e^{-a(t-\tau)(\eta^2+\zeta^2)}\,d\tau \tag{95}$$

The solution fo the initial equation (89) is obtained from equations (92) and (95) by means of Fourier's inversion theorem

$$T(y,z,t) = \frac{2\,a\sqrt{2}}{\pi^{3/2}} \int_0^\infty \int_0^\infty \int_0^t T_c^*(\eta,\tau)\,\zeta\, e^{-a(t-\tau)(\eta^2+\zeta^2)} \cos \eta y \sin \zeta z \, d\eta\, d\zeta\, d\tau \tag{96}$$

The integral involved in this solution is Fourier's sine transform of the function

$$\zeta e^{-a(t-\tau)\zeta^2}$$

$$G_1(z,\tau) = \sqrt{\frac{2}{\pi}} \int_0^\infty \zeta e^{-a(t-\tau)\zeta^2} \sin \zeta z d\zeta = \sqrt{\frac{2}{\pi}}\, \frac{\sqrt{\pi}\, z}{4[a(t-\tau)]^{3/2}}\, e^{-\frac{z^2}{4a(t-\tau)}}$$

The temperature function (96) may thus be written

$$T(y,z,t) = \frac{a\sqrt{2}}{2\pi} \int_0^t \int_0^\infty \frac{z e^{-\frac{z^2}{4a(t-\tau)}}}{a(t-\tau)^{3/2}}\; T_c^*(\eta,\tau)\, e^{-a(t-\tau)\eta^2} \cos \eta y \, d\eta\, d\tau \tag{97}$$

We shall determine the function $T_c^*(\eta,\tau)$ by the boundary conditions (91a)

$$T_c^*(\eta,\tau) = \sqrt{\frac{2}{\pi}} \int_0^\infty T(y,t) \cos \eta y \, dy$$

According to boundary condition (91a), the function $T(y,t)$ obtains the value $T(\tau)$ when $-b < y < +b$, and it is zero when $/y/ > b$. Consequently, we find by integration

$$T_c^*(\eta,\tau) = \sqrt{\frac{2}{\pi}}\, \frac{\sin \eta y}{\eta}\, T(\tau) \tag{98}$$

Inserting the expression (98) for T_c^* in equation (97), the temperature function is found to be

$$T(y,z,t) = \frac{a}{\pi^{3/2}} \int_0^t \int_0^\infty \frac{z e^{-\frac{z^2}{4a(t-\tau)}}}{[a(t-\tau)]^{3/2}}\, \frac{\sin \eta y}{\eta}\, T(\tau)\, e^{-a(t-\tau)\eta^2} \cos \eta y \, d\eta\, d\tau \tag{99}$$

The function $G_1(\eta, y) = \frac{\sin \eta y}{\eta}$ is Fourier's cosine transform of the function

$$g_1(\eta, y) \begin{cases} = \sqrt{\frac{\pi}{2}} & \text{when } -0 < y < +b \\ = \frac{1}{2}\sqrt{\frac{\pi}{2}} & \text{when } /y/ = b \\ = 0 & y = 0 \end{cases}$$

and $G_2(\eta, \tau) = e^{-a(t-\tau)\eta^2}$ is Fourier's cosine transform of

$$g_2(y, \tau) = \sqrt{\frac{2}{\pi}} \frac{\sqrt{\pi}}{2\sqrt{a(t-\tau)}} e^{-\frac{y^2}{4a(t-\tau)}}$$

The integral $\int_0^\infty G_1(\eta, y)\, G_2(\eta, y) \cos \eta y d\eta$ is Fourier's cosine transform of the product of two Fourier transforms. By convolution [7; p. 44], we obtain

$$\int_0^\infty G_1(\eta, y) G_2(\eta, y) \cos \eta y d\eta = \frac{1}{2} \int_0^\infty \sqrt{\frac{\pi}{2}} \frac{\sqrt{\pi}}{2\sqrt{a(t-\tau)}} \left[e^{-\frac{(y-v)^2}{4a(t-\tau)}} + e^{-\frac{(y+v)^2}{4a(t-\tau)}} \right] dv$$

Substituting the integral in equation (99), the solution is found to be

$$T(y, z, t) = \frac{a}{4\pi} \int_0^t \int_0^\infty \frac{z\, e^{-\frac{z^2}{4a(t-\tau)}}}{[a(t-\tau)]^2} \left[e^{-\frac{(y-v)^2}{4a(t-\tau)}} + e^{-\frac{(y+v)^2}{4a(t-\tau)}} \right] T(\tau) dv d\tau \qquad (100)$$

We shall further modify this solution for more convenient numerical calculation, writing

$$\frac{y-v}{\sqrt{4a(t-\tau)}} = \beta$$

$$\frac{y+v}{\sqrt{4a(t-\tau)}} = \gamma$$

which converts the inner integral into

$$\int_0^b \left[e^{-\frac{(y-v)^2}{4a(t-\tau)}} + e^{-\frac{(y+v)^2}{4a(t-\tau)}} \right] dv - \int_b^\infty [\,]\, dv =$$

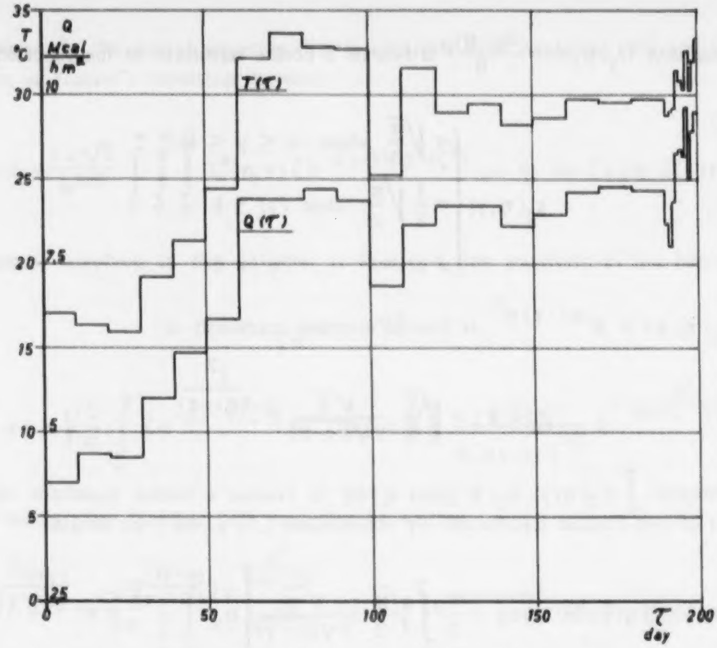

Fig. 13. Surface temperature of the slab, and thermal flow that has entered the ground as a function of time measured in one of the experimental houses. Starting point: 1st of September.

The integral from b to ∞ vanishes

$$-\sqrt{4a(t-\tau)}\int_{\frac{y}{\sqrt{4a(t-\tau)}}}^{\frac{y-b}{\sqrt{4a(t-\tau)}}} e^{-\beta^2}\,d\beta + \sqrt{4a(t-\tau)}\int_{\frac{y}{\sqrt{4a(t-\tau)}}}^{\frac{y+b}{\sqrt{4a(t-\tau)}}} e^{-\gamma^2}\,d\gamma =$$

$$\sqrt{\frac{\pi}{2}}\,\sqrt{4a(t-\tau)}\left[\operatorname{erf}\frac{y+b}{\sqrt{4a(t-\tau)}} - \operatorname{erf}\frac{y-b}{\sqrt{4a(t-\tau)}}\right]$$

Substituting the resulting expression of the integral in equation (100), we have

$$T(y,z,t) = \frac{a}{4\sqrt{\pi}}\int_0^t \frac{z\,e^{-\frac{z^2}{4a(t-\tau)}}}{[a(t-\tau)]^{3/2}}\left[\operatorname{erf}\frac{y+b}{\sqrt{4a(t-\tau)}} - \operatorname{erf}\frac{y-b}{\sqrt{4a(t-\tau)}}\right]T(\tau)\,d\tau \tag{101}$$

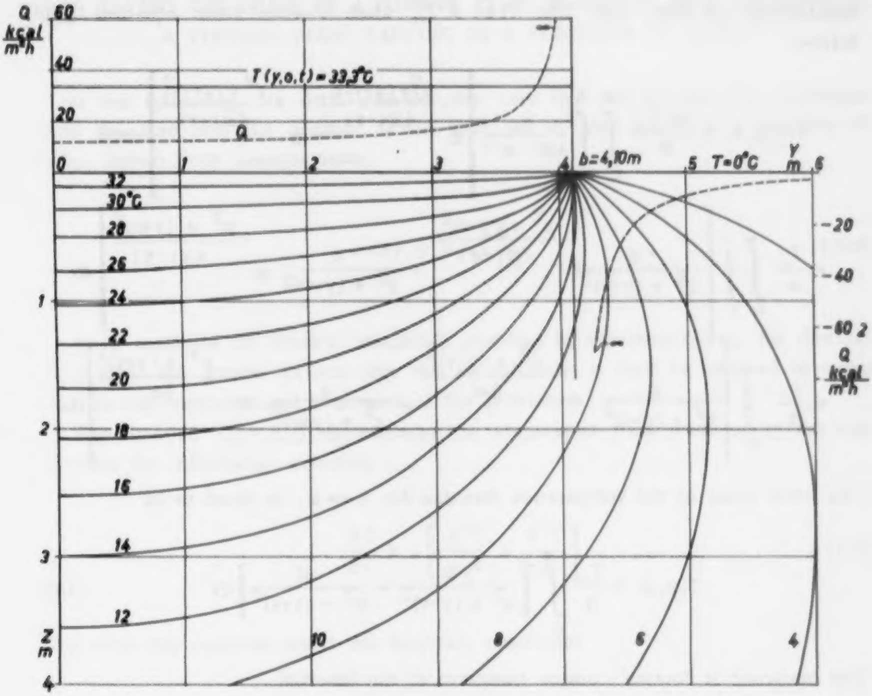

Fig. 14. Course of the isothermal lines under a foundation slab having the shape of a narrow strip, and its surface temperature at the time t = 200 days when its surface temperature varies as a function of time in conformity with Fig. 13; a = 0,0032 m^2/h.

Equation (101) represents the temperature, due to the effect from the slab, in the point (y, z) at the time t, when $z > 0$. When $T(\tau)$ is known (Fig. 13), the temperature field is obtained by means of equation (14) (Fig. 14).

The expression of the function reveals immediately that it satisfies the boundary conditions (90) and (91a), (91b) and (91c). It is also easy to compute with the aid of tables and graphical integration.

By the limit process $t \to \infty$, equation (39), p. 22, describing the temperature distribution in the stationary state, is obtained from equation (101). In order to show this, we shall return to equation (100) and integrate it with respect to t,

considering, at first, that $\lim T(\tau) = T_o$ when $t \to \infty$. From equation (100) follows

$$T(y,z,t) = \frac{4a\ T_o}{\pi} \int_0^\infty \int_0^t \frac{z}{4a(t-\tau)^2} \left[e^{-\frac{z^2+(y-v)^2}{4a(t-\tau)}} + e^{-\frac{z^2+(y+v)^2}{4a(t-\tau)}} \right] dv d\tau$$

$$= \frac{T_o}{\pi} \int_0^\infty \Bigg|_0^t \left[\frac{-z}{z^2+(y-v)^2} e^{-\frac{z^2+(y-v)^2}{4a(t-\tau)}} + \frac{-z}{z^2+(y+v)^2} e^{-\frac{z^2+(y+v)^2}{4a(t-\tau)}} \right] dv$$

$$= \frac{T_o}{\pi} \int_0^\infty \left[\frac{z}{z^2+(y-v)^2} e^{-\frac{z^2+(y-v)^2}{4at}} + \frac{z}{z^2+(y+v)^2} e^{-\frac{z^2+(y+v)^2}{4at}} \right] dv$$

The limit value of the temperature function for $t \to \alpha$ is found to be

$$T(y,z) = \frac{T_o}{\pi} \int_0^\infty \left[\frac{z}{z^2+(y-v)^2} + \frac{z}{z^2+(y+v)^2} \right] dv \tag{102}$$

The integrand is Fourier's cosine transform of the function

$$\sqrt{\frac{2}{\pi}}\,\pi \cos(y\eta)\, e^{-z\eta}$$

and the temperature function is

$$T(y,t) = \frac{T_o}{\pi} \sqrt{\frac{2}{\pi}}\ \frac{\pi\sqrt{2}}{\sqrt{\pi}} \int_0^\infty \int_0^\infty \cos(y\ \eta) \cos(v\eta)\, e^{-z\eta}\ d\eta dv$$

We integrate with respect to v

$$T(y,z) = \frac{2T_o}{\pi} \int_0^\infty \int_0^b \frac{\sin \eta v}{\eta} \cos y\eta\, e^{-z\eta}\ d\eta dv$$

$$T(y,t) = \frac{2T_o}{\pi} \int_0^\infty \frac{\sin \eta b}{\eta} \cos y\eta\, e^{-z\eta}\ d\eta$$

This is the same equation as (39) on page 22, if we replace y by x and η by ξ.

The thermal flow into the ground has been calculated from the isotherm field (Fig. 14).

SLAB HAVING THE SHAPE OF A NARROW STRIP; RELEASING INTO THE GROUND A THERMAL FLOW VARYING AS A FUNCTION OF TIME

In the following, we shall consider the case that the thermal flow conveyed from the slab into the ground, Q (in kcal per m^2 and hour), is a function of time, known from measurements.

$$Q(y,t) = -\lambda \frac{\partial T}{\partial z} \tag{103}$$

As a result of the thermal insulation provided by the snow cover, the thermal flow from the ground surface into the air is small; it shall be assumed to equal zero in the mathematical treatment of the problem.

The function $T_2(y,z,t)$ representing the temperature field produced by the slab satisfies the differential equation

$$\frac{\partial T}{\partial t} = a\left[\frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right] \tag{104}$$

We solve this equation under the boundary conditions

$$T(y,z,0) = 0 \tag{105}$$

and

$$\begin{aligned}
&\text{a)}\ \frac{\partial}{\partial z} T(y,0,t) = Q(y,t) = -\frac{1}{\lambda} Q(t) \quad \text{when} \quad -b < y < b \\
&\qquad\qquad\qquad\qquad\ = 0 \qquad\qquad\ \ \text{"} \qquad b < y < -b \\
&\text{b)}\ \frac{\partial}{\partial y} T(0,z,t) = 0 \\
&\text{c)}\ T(y,z,t) \rightarrow 0 \quad \text{when } y \text{ and } z \rightarrow \infty
\end{aligned} \tag{106}$$

In order to find the solution, we convert this equation into a linear, homogeneous differential equation with the aid of Fourier's double transformation

$$T^*(\eta,\zeta,t) = \frac{2}{\pi}\int_0^\infty\int_0^\infty T(y,z,t)\cos\eta y\cos\zeta z\,dy\,dz \tag{107}$$

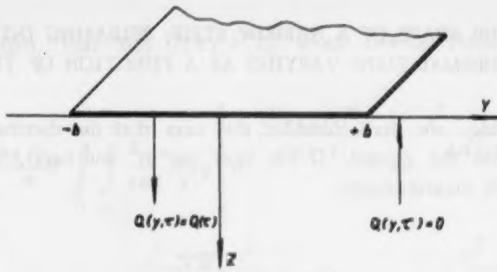

Fig. 15. Foundation slab having the shape of a narrow strip; the thermal flow entering the ground from the slab being a given function of time.

For this purpose, both members of equation (104) are multiplied by $\frac{2}{\pi}\cos\eta y\cos\zeta z$ and integrated, considering the boundary conditions

$$\frac{2}{\pi}\int_0^\infty\int_0^\infty \frac{\partial T}{\partial t}\cos\eta y\cos\zeta z\,dydz = \frac{2a}{\pi}\int_0^\infty\int_0^\infty\left(\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)\cos\eta y\cos\zeta z\,dy\,dz$$

$$\frac{2}{\pi}\int_0^\infty\int_0^\infty \frac{\partial T}{\partial t}\cos\eta y\cos\zeta z\,dydz = \frac{\partial T^*}{\partial t}$$

$$\frac{2a}{\pi}\int_0^\infty\int_0^\infty \frac{\partial^2 T}{\partial z^2}\cos\zeta z\cos\eta y\,dzdy = \frac{a\sqrt{2}}{\sqrt{\pi}}\sqrt{\frac{2}{\pi}}\int_0^\infty Q(y,t)\cos\eta y\,dy -$$

$$\frac{2a}{\pi}\zeta^2\int_0^\infty\int_0^\infty T(y,z,t)\cos\eta y\cos\zeta z\,dydz = \frac{a\sqrt{2}}{\sqrt{\pi}}Q^*(\eta,t) - a\zeta^2 T^*(\eta,\zeta,t)$$

$$\frac{2a}{\pi}\int_0^\infty\int_0^\infty \frac{\partial^2 T}{\partial y^2}\cos\eta y\cos\zeta z\,dydz = -\eta^2\frac{2a}{\pi}\int_0^\infty\int_0^\infty T(y,z,t)\cos\eta y\cos\zeta z\,dydz =$$

$$-a\eta^2 T(\eta,\zeta,t)$$

Equation (104) thus obtains the form of a linear, homogeneous differential equation of the first order

$$\frac{\partial T^*}{\partial t} + a(\eta^2 + \zeta^2)\, T^*(\eta, \zeta, t) - \frac{a\sqrt{2}}{\sqrt{\pi}} Q^*(\eta, t) = 0 \tag{108}$$

Its solution is

$$T^*(\eta, \zeta, t) = e^{-a(\eta^2+\zeta^2)\int_0^t dt}\left[T^*(\eta, \zeta, 0) + \frac{a\sqrt{2}}{\sqrt{\pi}}\int_0^t e^{a(\eta^2+\zeta^2)\int_0^\tau d\tau} Q^*(\eta, \tau)\, d\tau\right]$$

$$= T^*(\eta, \zeta, 0)\, e^{-at(\eta^2+\zeta^2)} + \frac{a\sqrt{2}}{\sqrt{\pi}}\int_0^t Q^*(\eta, \tau)\, e^{-a(t-\tau)(\eta^2+\zeta^2)}\, d\tau \tag{109}$$

According to boundary condition (105),

$$T^*(\eta, \zeta, 0) = \frac{2}{\pi}\int_0^\infty\int_0^\infty T(y, z, 0) \cos \eta y \cos \zeta z \, dydz = 0$$

The solution (109) of the linear equation becomes therefore

$$T^*(\eta, \zeta, t) = \frac{a\sqrt{2}}{\sqrt{\pi}}\int_0^t Q^*(\eta, \tau)\; e^{-a(t-\tau)(\eta^2+\zeta^2)}\, d\tau \tag{110}$$

The solution of the initial equation (104) is obtained from equations (107) and (110) according to Fourier's inversion theorem

$$T(y, z, t) = \frac{2a\sqrt{2}}{\pi^{3/2}}\int_0^\infty\int_0^\infty\int_0^t Q^*(\eta, \tau)\; e^{-a(t-\tau)(\eta^2+\zeta^2)} \cos \eta y \cos \zeta z \, d\eta\, d\zeta\, d\tau \tag{111}$$

The integral involved in this solution is Fourier's cosine transform of the function

$$e^{-a(t-\tau)\zeta^2}$$

$$G_1^*(z, \tau) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-a(t-\tau)\zeta^2} \cos \zeta z d\zeta = \sqrt{\frac{2}{\pi}}\,\frac{\sqrt{\pi}}{\sqrt{a(t-\tau)}}\, e^{-\frac{z^2}{4a(t-\tau)}}$$

The temperature function can thus be written

$$T(y,z,t) = \frac{a\sqrt{2}}{\pi} \int_0^t \int_0^\infty \frac{e^{-\frac{z^2}{4a(t-\tau)}}}{\sqrt{a(t-\tau)}} Q^*(\eta,\tau)\, e^{-a(t-\tau)\eta^2} \cos \eta y \, d\eta \, d\tau \tag{112}$$

We determine the function $Q^*(\eta,\tau)$ by the boundary conditions (106a)

$$Q^*(\eta,\tau) = \sqrt{\frac{2}{\pi}} \int_0^\infty Q(y,t) \cos \eta y \, dy$$

The function $Q(y,t)$ obtains the value $-\frac{1}{\lambda} Q(t)$ when $0 < y < +b$ and the value 0 when $/y/ > b$. This function has the Fourier transform

$$Q^*(\eta,\tau) = -\sqrt{\frac{2}{\pi}} \frac{\sin \eta y}{\eta} Q(y,\tau) = \frac{1}{\lambda} \sqrt{\frac{2}{\pi}} \frac{\sin \eta y}{\eta} Q(\tau)$$

The temperature function (112) can thus be written

$$T(y,z,t) = -\frac{2a}{\pi^{3/2}} \int_0^t \int_0^\infty \frac{e^{-\frac{z^2}{4a(t-\tau)}}}{\sqrt{a(t-\tau)}} \frac{\sin \eta y}{\eta} Q(\tau)\, e^{-a(t-\tau)\eta^2} \cos \eta y \, d\eta \, d\tau \tag{113}$$

The function $g_1(\eta, y) = \frac{\sin \eta y}{\eta}$ is Fourier's cosine transform of the function

$$g_1(\eta,y) \begin{cases} = \sqrt{\frac{\pi}{2}} & \text{when } 0 < y < +b \\ = \frac{1}{2}\sqrt{\frac{\pi}{2}} & \text{"} \quad y = b \\ = 0 & \text{"} \quad y > b \end{cases}$$

and $G_2(\eta, y) = e^{-a(t-\tau)\eta^2}$ is Fourier's cosine transform of

$$g_2(y,\tau) = \frac{\sqrt{\frac{2}{\pi}}\,\sqrt{\pi}}{2\sqrt{a(t-\tau)}} e^{-\frac{y^2}{4a(t-\tau)}}$$

The integral $\int_0^\infty G_1(\eta, y)G_2(\eta, y)\cos\,\eta y d\eta$ is Fourier's cosine transform of the product of two cosine transforms. By convolution, we obtain

$$\int_0^\infty G_1(\eta, y)G_2(\eta, y)\cos\,\eta y d\eta = \frac{1}{2}\int_0^\infty \sqrt{\frac{2}{\pi}}\,\frac{\sqrt{\pi}}{2\sqrt{a(t-\tau)}}\left[e^{-\frac{(y-v)^2}{4a(t-\tau)}}\quad e^{-\frac{(y+v)^2}{4a(t-\tau)}}\right]dv$$

Substituting the integral in equation (113), the solution is found to be

$$T(y,z,t) = \frac{a}{2\pi\lambda}\int_0^t\int_0^\infty \frac{e^{-\frac{z^2}{4a(t-\tau)}}}{a(t-\tau)}\left[e^{-\frac{(y-v)^2}{4a(t-\tau)}} + e^{-\frac{(y+v)^2}{4a(t-\tau)}}\right]Q(\tau)dvd\tau \tag{114}$$

We shall further modify this solution for more convenient calculation, writing

$$\frac{y-v}{\sqrt{4a(t-\tau)}} = \beta;$$

$$\frac{y+v}{\sqrt{4a(t-\tau)}} = \gamma;$$

which converts the inner integral into

$$\int_0^b \left[e^{-\frac{(y-v)^2}{4a(t-\tau)}} + e^{-\frac{(y+v)^2}{4a(t-\tau)}}\right]dv =$$

$$= -\sqrt{4a(t-\tau)}\int_{\frac{y}{\sqrt{4a(t-\tau)}}}^{\frac{y-b}{\sqrt{4a(t-\tau)}}} e^{-\beta^2}\,d\beta + \sqrt{4a(t-\tau)}\int_{\frac{y}{\sqrt{4a(t-\tau)}}}^{\frac{y+b}{\sqrt{4a(t-\tau)}}} e^{\gamma^2}\,d\gamma =$$

$$= \frac{\sqrt{\pi}}{2}\sqrt{4a(t-\tau)}\left[\operatorname{erf}\frac{y+b}{\sqrt{4a(t-\tau)}} - \operatorname{erf}\frac{y-b}{\sqrt{4a(t-\tau)}}\right]$$

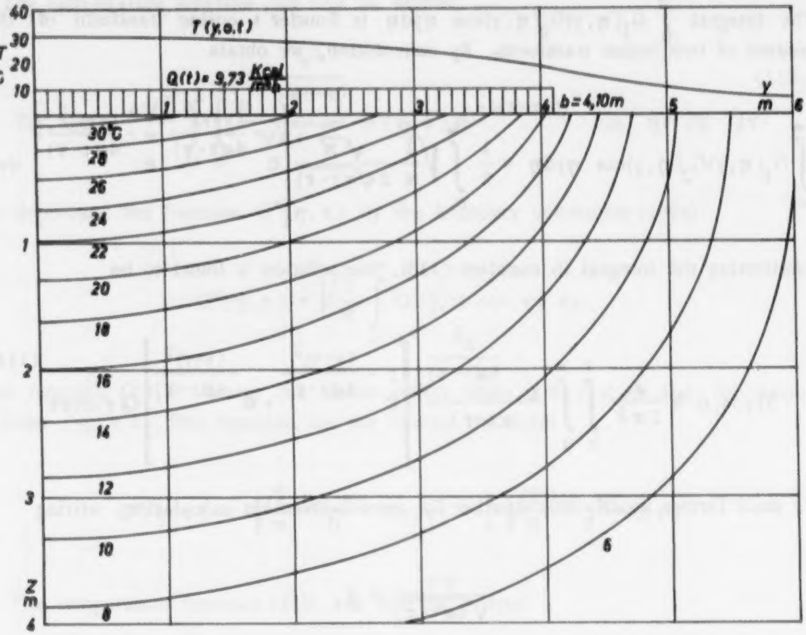

Fig. 16. Course of the isothermal lines under a foundation slab having the shape of a narrow strip, and its surface temperature as a function of y at the time t = 200 days when the thermal flow varies as a function of time in conformity with Fig. 13; $a = 0{,}0032\ m^2/h$, $\lambda = 1{,}00$ kcal/m h°C.

Substituting the resulting expression of the integral in equation (114), we have

$$T(y,z,t) = \frac{a}{\lambda\sqrt{\pi}} \int_0^t \frac{e^{-\frac{z^2}{4a(t-\tau)}}}{\sqrt{4a(t-\tau)}} \left[\operatorname{erf}\frac{y+b}{\sqrt{4a(t-\tau)}} - \operatorname{erf}\frac{y-b}{\sqrt{4a(t-\tau)}}\right] Q(\tau)d\tau \tag{115}$$

$Q(\tau)$ is known from measurements (Fig. 12). The temperature field produced by the slab is obtained from equation (115) (Figs. 16 and 17).

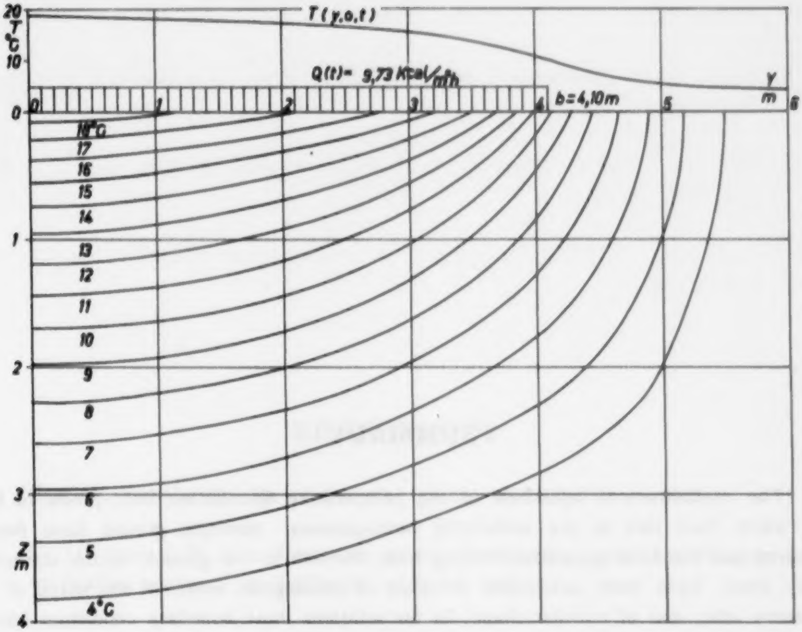

Fig. 17. Course of the isothermal lines under a foundation slab having the shape of a narrow strip, and its surface temperature as a function of y at the time t = 200 days when the thermal flow varies as a function of time in conformity with Fig. 13; $a = 0,00022\ m^2/h$, $\lambda = 1,43$ kcal/m h°C.

SUMMARY

The mathematical equations of the temperature distribution field produced by a warm floor slab in the underlying homogeneous, isotropic ground have been solved and the heat quantities flowing from the slab to the ground, in the stationary state, have been calculated for slabs of rectangular form, of the shape of a narrow strip, and of circular shape. In the solutions those boundary conditions have been found which are most appropriate for use in cases occurring in actual practice.

Non-stationary temperature fields for which the surface temperature of the slab or the thermal flow from the slab to the ground is known as a function of time are presented for a slab having the shape of a narrow strip. The equations can also be employed to calculate the field, varying with time, which exists in the vertical section in the middle of an elongated, rectangular slab. Fourier's and Hankel's transformations have been employed in the solutions and the unknown functions involved have been found by means of a systematic procedure, e.g., as solutions of dual integral equations.

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